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**КУРС ЛЕКЦИЙ**

**по дисциплине**

**«Моделирование процессов деформирования твердых тел (на английском)»**

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named after Professor Mustafin T.G.

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**THE COURSE OF LECTURES**

**on the discipline**

**«Modeling the processes of solids deformation (in English)»**

Educational program 7M05402 - Mechanics

Karaganda 2022

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**Introduction**

A wide range of problems and applications for mechanics stimulates the research and study of the basic principles for modeling the processes of deformation of solids.

The discipline "Modeling the processes of solids deformation (in English)" is offered to undergraduates under the educational program 7M05402 – Mechanics.

The purpose of studying the discipline "Modeling the processes of solids deformation (in English)" is to form undergraduates' knowledge about the basic principles and provisions for modeling the deformation processes of solids in problems of mechanics.

The course of lectures consists of an introduction, ten lectures and a list of recommended literature.

The structure of each lecture can be described as follows.

The lecture plan contains the main sections and points of the lecture material.

The content of the lecture presents the main provisions, assumptions, characteristics, criteria, properties, equations of the material presented. The material of the lectures is supplemented with figures with explanations.

To self-control the assimilation of the lecture material, undergraduates need to answer the questions given at the end of the lecture material.

The lecture ends with a list of recommended literature.

The list of recommended literature is given at the end of each lecture, which meets the requirements of the design of the course of lectures at the Buketov Karaganda University.

The course of lectures "Modeling the processes of solids deformation (in English)" involves the study of lectures by undergraduates in the following order.

It is necessary to familiarize with the content of the lecture, then to study the presented formulas and equations in order to understand the presented lecture material. If the lecture contains examples, then you should solve them in writing. And finally you should answer the questions for self-control.

It should be borne in mind that the course of lectures is not an original scientific research, does not pretend to provide exhaustive information, but aims to give students an idea of the main content of the discipline "Modeling the processes of solids deformation (in English)".

**Lecture 1**

**Lecture topic: Technical information on the modelling approach**

**The plan**

**1. The basic equations**

1.1. Continuity equation

1.2. Momentum equation

1.3. Energy equation

**2. Finite element implementation**

2.1. Application of rheology

2.2. Energy feedback effects

**1. The basic equations**

Computer modelling in mechanics was founded over three decades ago, providing a platform for the publication of papers in this important field of science and engineering. The range of appropriate contributions is very wide. It covers any type of computational method for the simulation of complex physical problems leading to the analysis and design of engineering products and systems. This includes theoretical development and rational applications of mathematical models, variational formulations, and numerical algorithms related to finite element, boundary element, finite difference, finite volume, and meshless discretization methods in the following fields of computational science and engineering:

• solid and structural mechanics

• fluid mechanics

• mechanics of materials

• heat transfer

• dynamics

• geomechanics

• acoustics

• biomechanics

• nanomechanics

• molecular dynamics

• quantum mechanics

• electromagnetics

and also includes virtual design, multiscale phenomena, from nanoscale to macroscale, multiphysics problems, parallel computing, optimization, probabilistic and stochastic approaches.

The novelty of our approach is the fact that the continuity, momentum and energy equations are fully coupled. This coupling nucleates shear zones and governs their evolution. This approach is fundamentally different to previous approaches, which embodied shear zone nucleation in a constitutive equation without considering the energy evolution within and outside the shear zone.

Our approach generates shear zones only through critical state in the energy fluxes, whereas other models are based on empirical laws for localisation that are unrelated to energy.

**1.1. *Continuity equation***

The continuity equation accounts for mass conservation:

, (1)

where

 is the density and **u** is the local material velocity vector of the volume under consideration;

**** is the nabla vectorial differential operator, thus the scalar (dot) product with the velocity vector defines the divergence of the velocity field.

Equation (1) incorporates time as a derivative, which is implicitly derived from the evolution of isentropic work to be discussed in conjunction with equation (1). In most approaches this density variation is neglected to construct a simplified (Boussinesq) non-energy dependent form of the continuity equation. In which case the continuity equation is simplified to:

. (2)

We define a symmetric strain-rate tensor:

, (3)

where

T defines the transposed of the velocity gradient **** and

**x** is the local displacement vector of a material particle within the Lagrangian reference volume.

We can now describe continuity in terms of strain rates:

, (4)

where

 is the total strain rate,

 is the elastic strain rate,

 is the plastic strain rate,

 is the creep strain rate.

Creep strain rate is simplified in our model to steady state power-law behaviour.

Note that this additive strain rate decomposition does not exclude volumetric changes. While we discuss only the relation to the continuity equation, an additive split can also be derived for the Helmholtz free energy densities thus supplying the first fundamental mathematical theory for finite strain elasto-plasticity (Fish and Shek, 2000). For the volumetric deformation, the only active component of the additive strain rate decomposition is elastic:

. (5)

**1.2. *Momentum equation***

The momentum equation classically describes translational and rotational equilibrium conditions of forces within an arbitrary volume:

, (6)

where

 is the divergence of the Cauchy stress tensor and

 is the body force.

The Cauchy stress tensor is defined as follow:

, (7)

where

 are the surface tractions,

**n** is the normal vector to the surface considered.

**1.3. *Energy equation***

A detailed discussion on the formulation of the energy equation is described in Regenauer-Lieb and Yuen (2003). In its simplified form (equation (8)), the equation incorporates a shear heating term, an isentropic power term and a component of energy diffusion due to heat conduction.

. (8)

The evolution of heat (**)** is defined as:

, (9)

where

 is the density,

is the specific heat and

 is the material derivative of the temperature change (Lagrangian framework).

Shear heating is the dissipated energy and is expressed as:

, (10)

where

 is the efficiency of converting mechanical work into heat ().

The shear heating efficiency of most materials is commonly 85% and 95% for large strain, meaning that almost all of the deformational work is converted into heat and very little is stored in microstructural defects. For simplicity we used a full conversion of mechanical work into heat (i.e. χ = 1).  is the deviatoric stress, which is defined by:

, (11)

where



is the trace of the Cauchy stress tensor, or the pressure. The scalar product of the stress and strain rate tensors in equation (10) has a physical meaning of a power in a unit volume (also called the strain energy rate density).

The isentropic power term in equation (8) is the elastic feedback term, which is the non-dissipative reversible elastic deformational work rate. It describes thermal-elastic effects such as dilation on heating and shrinking on cooling.

The term thus shows the pressure dependent energy fluxes due to adiabatic volume change:

, (12)

where

**** is the linear coefficient of thermal expansion and

*T*equ is the equilibrium temperature change of adiabatic expansion or compression.

The conduction term in equation (8) is a function of diffusivity ():

, (13)

where

**** is also known as the Laplacian scalar differential operator defined by the square of the nabla operator.

Inserting equations (9-13) in equation (8) gives the energy equation:

. (14)

**2. Finite element implementation**

Standard finite element approaches use a method of virtual displacements applied on the momentum equation to minimize the strain energy rate density () as a weak form of the momentum equation. In the elastic case, all the virtual displacements are recoverable and only stored energy appears in one time step. In the elasto-visco-plastic case, which we use, stored and dissipated energy occur within one time step.

While the classical finite element approach is designed to satisfy only the coupled momentum and continuity equations, our approach using the minimization of the stored and dissipated strain energy rate density functionals, can be formulated so as to comply with the second law of thermodynamics when applied to small displacements. Mathematically, this can be presented as:

, (15)

where

is the virtual velocity,

is the volume considered and

 is the surface.

For the implicit time integration of the minimization, we consider an adaptive time stepping technique. This ensures that in unstable processes, such as the nucleation of shear zones, the variational principle of least dissipative power will give post-bifurcational solutions controlled by the rates of the energy fluxes. These are thus self-consistent in terms of classical mechanics and are compatible in terms of non-equilibrium variational thermodynamics. This enables us to formulate criteria for numerically tractable solutions in the post-bifurcational state using the variational principles of finite element analyses and an adaptive time stepping scheme controlled by a critical thermodynamic state (Regenauer-Lieb and Yuen, 2004).

**2.1. *Application of rheology***

For ductile and brittle behaviour we use a comprehensive rheology by assuming von Mises plasticity, with a pressure dependent yield stress (*τ*) combined with the elasto-visco-plastic coaxial flow rule. Equation (4) now reads:

, (16)

where

*E* is Young’s modulus and *ν* is Poisson ratio,

 is the objective co-rotational stress rate,

**** is the Kronecker delta,

 and *n* are material constants,

 is the activation enthalpy,

*R* is the universal gas constant.

 is defined as the second invariant of the deviatoric stress tensor:

. (17)

The elastic strain rate (the first term in equation (16)) is divided into three terms: the first corresponds to the elastic shear strain rate, the second term corresponds to the elastic volumetric strain rate and the third term corresponds to the thermal expansion. The fourth term corresponds to plastic strain rate, where the scalar multiplier  is constrained by using the classical elastic-plastic flow rule. The fifth term is the classical power law for steady state creep, which applies for both olivine and quartz rheologies (Hirth and Kohlstedt, 2004; Kronenberg and Tullis, 1984). One can see that the third and the fourth terms do not comprise any permanent volumetric changes (see equation (5)). Any volumetric changes related to the recoverable elastic deformation are defined in the second and third terms of equation (16).

For the brittle yield stress we let the magnitude of the yield stress scale linearly with pressure, and the yield envelope Φ be controlled by the  envelope (Hill, 1950) thus obtaining a cone in the stress space which has the principal stresses as axes of a Cartesian coordinate system.

. (18)

This is a simplification of the classical brittle rheology, since a coaxial flow rule is assumed and no volumetric damage occurs. Such volumetric damage is normally parameterised by Mohr-Coulomb fracture failure envelope scaled through one-dimensional frictional experiment (i.e. Byerlee’s law). Such models use non-associative flow law where the stress and the strain-rate tensors are non-coaxial. This activates preferred planes for faulting. However, there is no unique way to consider energy out of the Mohr-Coulomb theory.

Therefore, in our model we use a mathematically simpler yet closed form with respect to the energy equation, where brittle localisation phenomena are only triggered by the effect of non-associativity induced through thermal elastic expansion. In other words, the thermal expansion leads to a local departure from the non-dilatant shear strain rate of viscous and plastic deformation thus inducing the same phenomenon as described in the constitutive Mohr-Coulomb theory.

We use a flow law of wet quartzite where 0.4% wt water has been added in a sealed capsule (Kronenberg and Tullis, 1984). For the mantle, wet olivine rheology has been used (Hirth and Kohlstedt, 2004). The values of material constants and of the activation enthalpy are listed in Table 1.

**Table 1.** Parameters used in the numerical model

|  |  |  |  |
| --- | --- | --- | --- |
| Parameter | Name | Value | Units |
|  | Shear heating efficiency | 1 | - |
|  | Thermal diffusivity | Quartz = 0.7 x 10-6  Olivine = 0.8 x 10-6 | m-2s-1 |
|  | Thermal expansion | 3 x 10-5 | K-1 |
|  | Specific heat | Quartz = 800  Olivine = 1000 | J/(kg\*K) |
|  | Density | Quartz = 2800  Olivine = 3300 | kg/m-3 |
|  | Poisson ratio | 0.25 | - |
| *E* | Young’s modulus | 4.5 x 1010 | Pa |
| *A* | Material constant - pre-exponential parameter | Quartz = 1.3 x 10-34  Olivine = 3.6 x 10-16 | Pa-ns-1 |
| *N* | Power-law exponent | Quartz = 4  Olivine = 3.5 | - |
| *H* | Activation enthalpy | Quartz = 135  Olivine = 480 | kJ/mol |
| *B* | Thickness of the radiogenic layer  (top of crust) | 10 | km |
| *Qs* | Surface heat flow | 60 and 70 | mW/m2 |
| *Qm* | Mantle heat flow | 30 | mW/m2 |

**2.2. *Energy feedback effects***

Ductile and brittle localisation processes are in principle related. While the ductile instability relies chiefly on temperature dependent perturbations, the brittle feedback flow localisation relies on pressure perturbations.

Consider an arbitrary heterogeneity in the temperature field. This heterogeneity has the potential to focus deformation through the exponential temperature dependency of the power law creep in the additive strain rate decomposition (equation (16)). Additionally, any non-linear creep rheology with a power law exponent larger than one has self-focusing potential (shear thinning flow). Any flow localisation creates more mechanical work, and in turn, leads to more shear heating (equation (10)), thus lubricating the emerging shear zone.

This has a similar effect on the isentropic work terms since the local temperature increase also produces thermal expansion, thus increases the pressure around the initial perturbation (equation (12)). All the feedback mechanisms are thus positive, and the only mechanism that inhibits runaway effects is the thermal diffusion by conduction (equation (13)).

**Questions for self-control**

1. What are the main equations that should be taken into account when modeling mechanics problems?

2. What is the physical meaning of the continuity equation?

3. What is the general form of the momentum equation?

4. In which paper is a detailed discussion of the formulation of the energy equation described?

5. How are energy feedback effects defined?

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**Lecture 2**

**Lecture topic: Numerical modelling of coupled transient phenomena. The finite element method**

**The plan**

**1. Problems of mechanics**

1.1. Solid mechanics

1.2. Diffusion

1.3. Advection - diffusion

1.4. Boundary conditions

**2. Numerical tools: the finite element method**

2.1. Introduction

2.2. Finite element method

2.3. Finite difference method

**1. Problems of mechanics**

The basic phenomena involved in thermo-hydro-mechanical processes in geomaterials have been described in the 1st chapter. The equations describing these phenomena are non-linear differential equations. Their solution can generally only be approximated thanks to numerical methods. This is the main subject of the present chapter. First the type of problems and the set of equations to be solved will be shortly recalled. Emphasis will be given on the non-linear contributions. In a second section, the mostly developed numerical method, i.e. the finite difference and the finite element methods will be discussed under the light of the problems to be treated. The iterative techniques allowing the solution of non linear equations will be described. A third section will be dedicated to the coupling terms and to their modelling. The question that rises then is How to model efficiently problems, which may differ highly from, the point of view of the time and length scales?

We are here interested in a number of different physical phenomena (Gens, 2001), including:

* The non-linear solid mechanics, and especially the soil, rock or concrete mechanics: We consider the relations between displacements, strains, stresses and forces within solids. The material behaviour is described by a constitutive model, which can take into account elastoplasticity or elasto-visco-plasticity. On the other hand, large transformations and large strains may lead to geometrical non-linearities.
* The fluid flow within porous media: Fluid can be a single phase of various natures (water, air, gas, oil, …) or it can be an association of two fluids, leading to unsaturated media (water and air, oil and gas, oil and water,…). In the second case, partial saturation is leading to permeability and storage terms depending on the saturation degree or on the suction level, involving non-linear aspects.
* The thermal transfers within porous media. Conduction is the leading process in solid (in the geomaterial matrix), but convection can also occur in the porous volume, as a consequence of the fluid flow. Radiation transfer could also occur inside the pores, but it will be neglected here. Conduction coefficients and latent heat may depend on the temperature.
* The pollutant transport or any spatial transfer of substance thanks to the fluid flow. The pollutant concentration may be high enough to modify the densities, involving non-linear effects.

All these problems are non-linear ones, and can be formulated with sets of partial differential equations. Moreover only three types of differential equations have to be considered, concerning respectively i) solid mechanics, ii) diffusion and iii) advection-diffusion problems.

**1.1. *Solid mechanics***

On the one hand, solid mechanics can be modelled on the following basis. The equilibrium equation is:

, (1)

where

*P* is the vector of volume forces,

σ is the Cauchy’s stress tensor and ∂ represents the spatial partial derivative operator:

. (2)

The stress tensor is obtained thanks to the time integration of a (elastic, elastoplastic or elasto-visco-plastic) constitutive equation (Laloui, 2001; Coussy, 2001):

, (3)

where

 is the stress rate, *D* is the strain rate and

*k* is a set of history parameters (state variables, like e.g. the preconsolidation stress).

In the most classical case of elastoplasticity, this equation reduces to :

. (4)

Most constitutive equations for geomaterials are non-linear ones.

When modelling a solid mechanics problem with the finite element method, the most used formulation is based on displacements *u* or on actualised coordinates *x*. If one considers only small strains and small displacements, the strain rate reduces to the well-known Cauchy’s strain rate :

. (5)

However, if large strains are to be considered, the preceding equations have to be reconsidered. The stress – strain rate couple has to be more precisely defined, with respect to the configuration evolution. Among multiple other choices (cf. Piola-Kirchoff stress – Green strain), we will only consider here the Cauchy’s stress and the Cauchy’s strain rate.

These tensors are defined in global axis in the current configuration, which is continuously deforming. If we note *X* the coordinates in a reference state[[1]](#footnote-1), and *x* the coordinates in the current configuration (figure 1), we can define the Jacobian tensor of the transformation:

. (6)

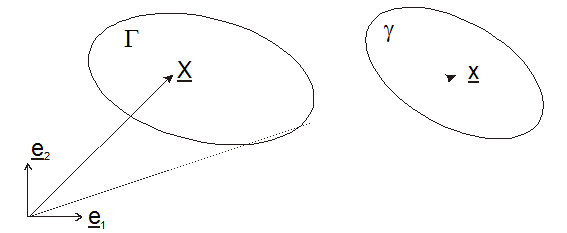


Figure 1. Initial and current configurations

The velocity gradient is defined as :

. (7)

The symmetric part of the velocity gradient is the strain rate associated to the Cauchy’s stress :

. (8)

The material stress evolution must then be described as a function of the strain rate, thanks to a constitutive equation [3]. This subject is described in other chapters of this book. However, as the Cauchy’s stress tensor is defined in global axis, the solid rotations will modify the tensor Cartesian components.

This evolution is not linked to strains and so is not described by the constitutive equation. Among other possibilities, the Jaumann’s objective derivative of the stress is a good component update:

, (9)

. (10)

Such large strain model [6-10] is non-linear.

Time dimension is not to be addressed for solid mechanics problems, unless when viscous term are considered in the constitutive model. Generally, the time that appears in the time derivatives in [3, 4, 5, 7-10] is only a formal one.

**1.2. *Diffusion***

Fluid flow in porous media and thermal conduction exchanges in solids are modelled thanks to similar diffusion equations.

The balance equation writes:

, (11)

where *f* represents a flux of fluid or heat, *Q* represents a sink term and *S* represents the storage of fluid or of heat. When modelling a diffusion problem with the finite element method, the most used formulation is based on fluid pore pressure *p* or on temperature *T*.

Then the Darcy’s law for fluid flow in porous media gives the fluid flux :

 (12)

with the intrinsic permeability *k* (possibly depending on the saturation degree), the dynamic viscosity *μ*, the density *ρ*, and the gravity acceleration *g*. The fluid storage term depends on the saturation degree *Sr* and on the fluid pressure

. (13)

For thermal conduction one obtains the Fourier’s law

 (14)

with the conductivity coefficient *λ*. The heat storage (enthalpy) term depends on the temperature

. (15)

The diffusion problem is non linear when:

* the permeability depends (directly or indirectly) on the fluid pore pressure,
* the fluid storage is a non-linear function of the pore pressure,
* partial saturation occurs,
* the conductivity coefficient depends on the temperature,
* the enthalpy is a non-linear function of the temperature.

When the storage term is considered, time dimension of the problem has to be addressed.

**1.3. *Advection – diffusion***

Transport of pollutant or of heat in porous media is governed by a combination of advection and diffusion (Thomas, 2001). Advection phenomenon is related to the transport (noted as a flow *fadv*) of an substance by a fluid flow, described by its velocity *fdifffluid*:

. (16)



The substance concentration *C* is generally supposed to be small enough not to influence the fluid flow. In porous media, due to the pores network tortuosity and to the friction, advection is always associated to a diffusion characterized by diffusion – dispersion tensor *D*. Therefore, the total flux of substance is:

. (17)

Balance equation and storage equations may be written in a similar way to the one for diffusion problems [11, 13, 15].

Compared to diffusion constitutive law [12,14], it appears here an advection term which doesn’t depend on the concentration gradient, but directly on the concentration. This is modifying completely the nature of the equations to be solved. Problems dominated by advection are very difficult to solve numerically (Charlier and Radu, 2001). In order to evaluate the relative advection effect, it is useful to evaluate the Peclet’s number, which is a ratio between diffusive and advective effects :

, (18)

where *h* is an element dimension.

**1.4. *Boundary conditions***

In the preceding section, differential equations were given for three types of problems. Solving these equations needs to define boundary and initial conditions. Classical boundary conditions may be considered : imposed displacements or forces for solid mechanics problems, imposed fluid pressures / temperatures / concentrations or imposed fluxes for diffusion and advection – diffusion problems.

However, it may be useful to consider much complex boundary conditions. For example, in solid mechanics, unilateral contact with friction or interface behaviour is often to be considered.

When coupling phenomena, the question of boundary conditions rises in complexity and has to be discussed.

On the other hand, initial conditions are often difficult to determine in geomechanics. Consider for example the problem of initial stress state. A long discussion may be proposed with respect for example to the long-term tectonic effect (Barnichon, 1998).

**2. Numerical tools: the finite element method**

**2.1. *Introduction***

An approximated solution of most problems described by a set of partial differential equations may be obtained thanks to numerical method like the finite element method (FEM), the discrete element method (DEM), the finite difference method (FDM), the finite volume method (FVM), or the boundary element method (BEM). For the problems concerned here, the most used methods are the finite element one and the finite difference one.

Non-linear solid mechanics is better solved thanks to the finite element method. Boundary element methods have strong limitation in the non-linear field. Finite difference methods are not easy to apply to tensorial equations (with the exception of the FLAC code, developed by Itasca).

Diffusion and advection – diffusion problems are often solved by finite difference or finite element method. Some finite difference codes are very popular for fluid flow, like e.g. MODFLOW for aquifer modelling or ECLIPSE (Eclipse, 2000) for oil reservoir modelling. These codes have been developed for a number of years and posses a number of specific features allowing taking numerous effects into account. However, they suffer from some drawbacks, which limit their potentialities for modelling coupled phenomena. Therefore we will only give little information about finite differences.

**2.2. *Finite element method***

The basic idea of the finite element method is to divide the field to be analysed into sub-domains, the so-called finite elements, of simple shape: e.g. triangles, quadrilaterals with linear, parabolic, cubic sides for two-dimensional analysis. In each finite element, an analytical simple equation is postulated for the variable to be determined, i.e. the coordinate or displacement for solid mechanics, and the fluid pressure, temperature, concentration for diffusion problems. In order to obtain C0  continuity, the unknown variable field has to be continuous at the limit between finite elements. This requirement is obtained thanks to common values of the field at specific points, the so-called nodes, which are linking the finite elements together. The field values at nodal points are the discretised problem unknowns.

For most solid mechanics and diffusion problems, isoparametric finite elements seem to be optimal (Zienckiewicz et all, 1989). The unknown field x may then be written, for solid mechanics cases[[2]](#footnote-2):

. (19)

It depends on the nodal unknowns xL and on shape functions NL, themselves depending on isoparametric coordinates ξ, η defined on a reference normalised space. Then the strain rate and the spin may be derived thanks to equations [8] and [10], the stress rate is obtained by [3], [4] and [9] and is time integrated. Eventually, equilibrium [1] has to be checked.

For scalar diffusion or advection – diffusion problems, the unknown field p (we will use hereafter the pore pressure notation, however temperature T or concentration C could be also considered changing the notation) may then be written:

 . (20)

It depends on the nodal unknowns pL and on shape functions NL. Then the fluid Darcy’s velocity and the storage evolution may be derived thanks to equations [12] and [13] (respectively [14-15] or [16-17]). No time integration is required here. Eventually, balance equation [11] has to be checked (§2.4).

The finite element method allows an accurate modelling of the boundary condition, thanks to easily adapted finite element shape. Internal boundaries of any shape between different geological layers or different solids can be modelled. Specific finite elements for interfaces behaviour or for unilateral boundaries may have also been developed (e.g. Charlier et all, 1990). Variations of the finite element size and density over the mesh are also easy to manage thanks to present mesh generators.

**2.3. *Finite difference method***

The finite difference method doesn’t postulate explicitly any specific shape of the unknown field. As we are concerned with partial differential equations, exact derivative are replaced by an approximation based on neighbour values of the unknown:

, (21)

where

the subscript *i* denotes the cell number and

*h* denotes the cell size.

For an orthogonal mesh, such derivatives are easily generalised to variable cell dimensions. However non-orthogonal meshes are asking question highly difficult to solve and are generally not used. Boundary conditions have then to be modelled by the juxtaposition of orthogonal cells, giving a kind of stairs for oblique or curved boundaries. Similarly, local refinement of the mesh induces irreducible global refinement. These aspects are the most prominent drawbacks of the finite difference method compared to the finite element one. On the other hand computing time is generally much lower with finite differences then with finite elements.

**Questions for self-control**

1. How are boundary conditions determined for mechanics problems?

2. What numerical methods are used in solving problems of mechanics?

3. What are the differences between the finite element method and the finite difference method?

4. What is the algorithm of the finite element method?

5. What is the algorithm for applying the finite difference method?

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**Lecture 3**

**Lecture topic: Numerical modelling of coupled transient phenomena. The Newton Raphson method. Transient effects**

**The plan**

**1. Numerical tools. The Newton Raphson method**

1.1. Solving the non-linear problem – the Newton Raphson method

1.2. The stiffness matrix

**2. Transient effects: the time dimension**

2.1. Time integration – diffusion problems

2.2. Time integration – solid mechanics

2.3. Scheme accuracy

**1. Numerical tools. The Newton Raphson method**

**1.1. *Solving the no- linear problem – the Newton Raphson method***

Let us now concentrate on the finite element method. The fundamental equation to be solved is the equilibrium equation (1) (respectively the balance equation for diffusion phenomena). As the numerical methods are giving an approximated solution, the equilibrium / balance equation has to be solved with the best compromise. This is obtained thanks to a global weak form of the local equation. Using weighted residuals, for solid mechanics, one obtains:

. (1)

And for diffusion phenomena:

, (2)

where

 and *q* are surface terms of imposed loads / fluxes,

The weighting functions are denoted *δu* and *δp,*

*δε* represents a derivative of the weighting function based on the Cauchy's strain derivate operator.

An equivalent equation could be obtained based on the virtual power principle. The *δu* and *δp* would then be interpreted as virtual arbitrary displacements and pressures. Within the finite element method, these global equilibrium / balance equation will be verified for a number of fundamental cases equivalent to the degrees of freedom (dof) of the problem, i.e. the number of nodes times the number of freedom degrees per node, minus to imposed values. The corresponding weighting functions will have simple forms based on the element shape functions[[3]](#footnote-3).

Giving a field of stress or of flux, using the weighting functions, one will obtain a value for each doff, which is equivalent to a nodal expression of the equilibrium / balance equation.

More precisely, for solid mechanics problems, one will obtain internal forces equivalent to stresses :

, (3)

where

*B* is a matrix of derivatives of the shape functions *N*.

If equilibrium is respected from the discretized point of view, these internal forces are equal to external forces (if external forces are distributed, a weighting is necessary):

. (4)

Similarly, for diffusion phenomena the nodal internal fluxes are equivalent to the local fluxes:

. (5)

If the balance equation is respected from the discretised point of view, these internal fluxes are equal to external ones:

. (6)

However, as we are considering non linear-problems, equilibrium / balance cannot be obtained immediately, but needs to iterate. This means that the equations are not fulfilled until the last iteration of each step.

Non-linear problems are solved for some decades, and different methods have been used. From our point of view, the Newton – Raphson is the reference method and probably the best one for a large number of problems. Let us describe the method. In the equation the internal forces *Fint* are depending on the basic unknown of the problem, i.e. the displacement field. Similarly in equation the internal fluxes are depending on the pressure (temperature, concentration, …) field.

If they don’t equilibrate the external forces / fluxes, the question to be treated can be formulated under the following form:

*How should we modify the displacement field (the pressure field) in order to improve the equilibrium (the balance) as stated by equation ?*

Following the Newton – Raphson method, one develops the internal force as a first order Taylor's series around the last approximation of the displacement field:

. (7)

This is a linearisation of the non-linear equilibrium equation. It allows obtaining a correction of the displacement field:

. (8)

The matrix noted *KLi,Kj* is the so-called *stiffness matrix*. With the corrected displacement field, one may evaluate new strain rates, new stress rates, and new improved internal forces. Equilibrium should then be improved.

The same meaning may be developed for diffusion problems: Taylor's development of the internal fluxes with respect to the pressures / temperatures / concentrations nodal unknowns.

The iterative process may be summarised as shown on figure 2 for one-dof solid mechanics problem. Starting from a first approximation of the displacement field *u(1)* one compute the internal forces *Fint(1)* (point *A(1)*) that are lower then the imposed external forces *Fext*. Equilibrium is then not fulfilled and a new approximation of the displacement field is searched. The tangent stiffness matrix is evaluated and an improved displacement is obtained *u(2)* (point *B(1)*). One computes again the internal forces *Fint(2)* (point *A(2)*) that are again lower then the external forces *Fext*. As equilibrium is not yet fulfilled, a new approximation of the displacement field is searched *u(3)* (point *B(2)*). The procedure has to be repeated until the equilibrium / balance equation is fulfilled with a given accuracy (numerical convergence norm). The process has a quadratic convergence, which is generally considered as the optimum numerical solution.

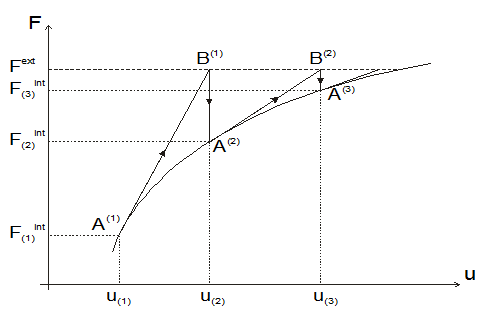


Figure 2. Illustration of the Newton – Raphson process

However the Newton – Raphson method has an important drawback : it needs important work to be developed as well as to be run on a computer. Especially the stiffness matrix *K*  is time consuming for the analytical development and for the numerical inversion. Therefore other methods have been proposed :

* Approximate stiffness matrix, in which some non-linear terms are neglected.
* Successive use of the same stiffness matrix avoiding new computation and inversion at each iteration

It should be noted that each alternative is reducing the numerical convergence rate. For some highly non-linear problems, the convergence may be loosed, and then no numerical solution may be obtained.

Some other authors, considering the properties and the efficiency of explicit time schemes in rapid dynamic (like for shocks modelling) add an artificial mass to the problem in order to solve it as a quick dynamic one. It should be clear that such technique might degrade the accuracy of the solution, as artificial inertial effects are added and the static equilibrium equation is not checked.

**1.2. *The stiffness matrix***

From equation (8), it appears that the stiffness matrix is a derivative of the internal forces:

. (9)

Two contributions will be obtained. On the one hand, one has to derive the stress state with respect with the strain field, itself depending on the displacement field. On the other hand, the integral is performed on the volume and the *B* matrix depends on the geometry. If we are concerned with large strains and if we are using the Cauchy's stresses, geometry is defined in the current configuration, which is changing from step to step, and even from one iteration to the other. These two contributions, the material one, issued from the constitutive model, and the geometric one have to be accurately computed in order to guaranty the quadratic convergence rate.

A similar discussion may be given for diffusive problems. However, the geometry is not modified for pure diffuse problems, so only the material term is to be considered.

**2. Transient effects: the time dimension**

Time dimension appears in first order time derivative in the constitutive mechanical model (3,9) and in the diffusion problems though the storage term. We will here discuss the time integration procedure and the accuracy and stability problems that are involved.

**2.1. *Time integration – diffusion problems***

The period to be considered is divided in time steps. Linear evolution of the basic variable with respect to the time is generally considered within a time step:

, (10)

where the subscripts A,B denote respectively the beginning and the end of a time step.

Then the pressure rate is:

 (11)

This time discretization is equivalent to a finite difference scheme. It allows evaluating any variable at any time within a time step.

The balance equation should ideally be fulfilled at any time during any time step. Of course this is not possible for a discretised problem. Only a mean assessment of the balance equation can be obtained. Weighted residual formulations have been proposed in a similar way as for finite elements (Zienckiewicz et al. 1989). However the implementation complexity is too high with respect with the accuracy. Then the easiest solution is to assess only the balance equation at a given time noted *θ* inside the time step:

. (12)

All variables have then to be evaluated at the reference time *θ* . Different classical schemes have been discussed for some decades:

* Fully explicit scheme - *θ =0* : all variables and the balance are expressed at the time step beginning, where everything is known (solution of the preceding time step). The solution is therefore apparently very easy to be obtained.
* Crank-Nicholson scheme or mid-point scheme - *θ =1/2*
* Galerkin's scheme - *θ =2/3*
* Fully implicit scheme - *θ =1*

The last three schemes are function of the pore pressure / temperature / concentration at the end of the time step, and may need to iterate if non-linear problems are considered.

For some problems, phase changes or similar large variations of properties may occur abruptly. For example, icing or vaporising of water is associated to latent heat consummation and abrupt change of specific heat and thermal conductivity. Such rapid change is not easy to model. The change in specific heat may be smoothed using an enthalpy formulation, because enthalpy *H* is an integral of the specific heat *c*. Then finite difference of the enthalpy evaluated over the whole time step gives a mean value  and so allows accurate balance equation:

, (13)

. (14)

**2.2. *Time integration – solid mechanics***

For solid mechanics problems, the constitutive law form (3,4) is an incremental one at the difference with the ones for diffusion problems (12). The knowledge of the stress tensor at any time implies to have time integrated the constitutive law. The stress tensor is a state variable that is stored and transmitted from step to step based on its final / initial value, and this value plays a key role in the numerical algorithm.

Then, in quite all finite element code devoted to modelling, equilibrium is expressed at the end of the time steps, following then a fully implicit scheme - *θ =1,* and using the end of step stress tensor value.

However, integrating the stress history with enough accuracy is crucial for the numerical process stability and global accuracy. Integrating the first order differential equation:

 (15)

can be based on similar concepts as the one described in the preceding paragraph. Various time schemes based on different *θ* values may be used. Stability and accuracy discussion are similar.

When performing large time steps, obtaining enough accuracy may require to use sub-stepping: within each global time step (as regulated by the global numerical convergence and accuracy problem), the stress integration is performed at each finite element integration point after division of the step in a number a sub-steps allowing high accuracy and stability.

**2.3. *Scheme accuracy***

The theoretical analysis of a time integration scheme accuracy and stability is generally based on a simplified problem (Zienckiewicz et al, 1989). Let consider diffusion phenomena restricted to linear case. Introducing the discretized field into the constitutive equations gives for the Darcy law (neglecting here the gravity term for the sake of simplicity):

. (16)

Similarly the storage law (linear case) gives:

, (17)

where *c* is a storage parameter.

Neglecting source terms, the weak form of the balance equation writes then:

 (18)

Considering that nodal values are not concerned by the integration, it comes :

 (19)

which is valid for any arbitrary perturbation *δp*. Then:

, (20)

which is a simple system of linear equations with a time derivative, a storage matrix *C* and a permeability matrix *K*. One can extract eigenvalues of this system and so arrive to a series of scalar independent equations of similar form:

 (no summation), (21)

where L represents now the number of the eigenmode with the eigenvalue *αL* and will not be noted in the following.

The exact solution for equation (21) is a decreasing exponential:

. (22)

This problem represents then the damping of a perturbation for a given eignemode. Numerically, the modelling is approximated and numerical errors always appear. If the equation is well modelled, any numerical error will be rapidly damped, if the error source is not maintained. Following this analysis, the whole accuracy and stability discussion may be given on these last scalar equations (21, 23).

Introducing the time discretisation (11,12) in (21) gives:

, (23)

which allows to evaluate the end of step pressure as a function of the beginning of step one:

 (24)

with the amplification factor :

. (25)

To ensure the damping process of the numerical algorithm, which is the *stability condition*, it is strictly necessary that the amplification factor remains lower then unity:

. (26)

This condition is always verified if *θ ≥ ½* , and conditionally satisfied otherwise:

 if *θ < ½*. (27)

This last equation is not easy to verify, as it depends on the eingenvalues, which are generally not computed. Therefore, for classical diffusion process considered in geomaterials, the condition *θ ≥ ½* is generally used.

It should be noted that the amplification factor becomes negative for large time steps, unless for the fully implicit scheme. Then the pertubated pressure decreases monotically in amplitude but with changes of sign. This may be questionable for some coupled phenomena, as it could induce oscillation of the coupled problem.

Let us now consider the accuracy of the numerical schemes. Developing in Taylor's series the exact and numerical solution allows to compare them:

 (28)

It appears that the only Crank-Nicholson scheme

θ = ½

has second order accuracy properties. However this conclusion is limited to infinitesimal time steps. For larger time steps, as in most numerical models, the Galerkin's scheme

θ = 2/3

gives the optimal compromise and should be generally used.

The whole discussion related to the stability and accuracy of the proposed time numerical schemes was based on eigenmodes of a linear problem. Can we extrapolate them to general problems? The eigenvalue passage is only a mathematical tool to be able to consider scalar problems, and has no influence on our conclusions.

Oppositely, the non-linear aspects could modify sometimes our conclusions. However, it is impossible to develop the analysis for a general non-linear problem, and the preceding conclusions should be adopted as guidelines, as they appear to be fruitful in most cases.

**Questions for self-control**

1. What is the Newton Raphson method?

2. What is the physical meaning of the stiffness matrix?

3. How is the stiffness matrix calculated?

4. How is time integration performed in diffusion problems?

5. What are transient effects?

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**Lecture 4**

**Lecture topic: Finite element modelling: monolithical and staggered approaches**

**The plan**

**1. Advection diffusion processes**

**2. Finite element modelling: monolithical and staggered approaches**

2.1. Finite element modelling: monolithical approach

2.2. Physical aspects: various terms of coupling

2.2.1 Hydromechanical coupling

2.2.2. Two fluids flow in rigid porous media coupling

2.2.3. Diffusion and transport coupling

2.2.4. Thermo-hydro-mechanical coupling

2.3. Finite element modelling: staggered approach

**1. Advection diffusion processes**

Let us first consider a purely advective process. Then the transport is governed by the advection equation and by the balance equation. Associating these two equations, on obtains:

, (1)

which is a hyperbolic differential equation. It cannot be solved by the finite element or finite difference problem, but by characteristic methods. The idea is to follow the movement of a pollutant particle by simply integrating step by step the fluid velocity field. This integration has to be accurate enough, as errors are cumulated from one step to the next.

On the other hand, if advection is very small compared to diffusion, then the finite element and finite difference methods are really efficient.

For most practical cases, an intermediate situation holds. It can be checked by the Peclet's number, which is high for mainly advective processes and low for mainly diffusive one. As diffusion has to be taken into account, the numerical solution must be based on the finite element method (the finite difference one may also be used but will not be discussed here). However, numerical experiments show that the classical Galerkin's formulation gives very poor results with high spatial oscillations and artificial dispersion. Then new solutions have been proposed (Zienckiewicz and Taylor, 1989, Charlier and Radu, 2001). A first solution is based on the use in the weighted residual method of weighting function that differs from the shape one by an upwind term, i.e. a term depending in amplitude and direction on the fluid velocity field. The main advantage of this method is to maintain the finite element code formalism. However, it is never possible to obtain a highly accurate procedure. Numerical dispersion will always occur.

Other solutions are based on the association of the characteristic method for the advection part of the process and of the finite element method for the diffusive part (Li et al 1997). The characteristic method may be embedded in the finite element code, what has a strong influence on the finite element code structure. It is also possible to manage the two methods in separated codes, as in a staggered procedure.

**2. Finite element modelling: monolithical and staggered approaches**

**2.1. *Finite element modelling: monolithical approach***

Modelling the coupling between different phenomena should imply to model each of them and, simultaneously, all the interactions between them. A first approach consists in developing new finite elements and constitutive laws especially dedicated to the physical coupled problem to be modelled. This approach allows taking accurately all the coupling terms into account. However there are some drawbacks that will be discussed in a later section.

Constitutive equations for coupled phenomena will be shortly discussed in the following sections.

The number of basic unknowns and following the number of degrees of freedom – dof per node are increased. This has a direct effect on the computer time used for solving the equation system (up to the third power of the total dof number). Coupled problems are highly time consuming.

Isoparametric finite element will often be considered. However some specific difficulties may be encountered for specific problems. Nodal forces or fluxes are computed in the same way as for decoupled problems. However stiffness matrix evaluation is much more complex, as interactions between the different phenomena are to be taken into account. Remember that the stiffness or iteration matrix is the derivative of internal nodal forces / fluxes with respect to the nodal unknowns (displacements / pressures / …). The complexity is illustrated by the following scheme of the stiffness matrix, restricted to the coupling between two problems:

|  |  |
| --- | --- |
| Derivative of problem  **1** nodal forces with respect  to problem  **1** nodal unknowns | Derivative of problem  **1** nodal forces with respect  to problem  **2** nodal unknowns |
| Derivative of problem  **2** nodal forces with respect  to problem  **1** nodal unknowns | Derivative of problem  **2** nodal forces with respect  to problem  **2** nodal unknowns |

The part of the stiffness matrix in cells 1-1 and 2-2 are similar or simpler to the ones involved in uncoupled problems. The two other cells 1-2 and 2-1 are new and may be of certain complexity.

Remember also that the derivative consider internal nodal forces / fluxes as obtained numerically, i.e. taking into account all numerical integration / derivation procedures. On the other hand, large difference of orders of magnitude between different terms may cause troubles in solving the problem and so need to be checked.

Numerical convergence of the Newton – Raphson process has to be evaluated carefully. It is generally based on some norms of the out-of-balance forces / fluxes. However, coupling implies often mixing of different kinds of dof, which may not be compared without precaution. Convergence has to be obtained for each basic problem modelled, not only for one, which would then predominate in the computed indicator.

**2.2. *Physical aspects: various terms of coupling***

A large number of different phenomena may be coupled. It is impossible to discuss here all potential terms of coupling, and we will restrict ourselves to some basic cases often implied in environmental geomaterial mechanics. In the following paragraphs, some fundamental aspects of potential coupling are briefly described. More information can be obtained in dedicated chapters of this journal or specialised books.

**2.2.1. *Hydromechanical coupling***

Number of dof per node: 3 (2 displacements + 1 pore pressure) for 2D analysis and 4 (3 displacements + 1 pore pressure) for 3D analysis.

Coupling mechanical deformation of soils or rock mass and water flow in pores is a frequent problem in geomechanics. The first coupling terms are related to the influence of pore pressure on mechanical equilibrium through the Terzaghi's postulate

 (2)

with the effective stress tensor *σ’* related to the strain rate tensor thanks to the constitutive equation, and the unity tensor *I.*

The second type of coupling concerns the influence of the solid mechanics behaviour on the flow process, which comes first through the storage term.

Storage of water in saturated media is mainly due to pores strains, i.e. to volumetric changes in soil / rock matrix:

. (3)

Another effect, which may be considered, is the permeability change related to the pore volume change, which may for example be modelled by the Kozeni – Carman law as a function of the porosity

*k = k(n)*.

Biot proposed an alternative formulation for rocks where contacts between grains are much more important then in soils. Following Biot, the coupling between flow and solid mechanics are much more important (Detournay, 1991, Thimus – proc. Biot Conf.).

The time dimension may cause some problems. First implicit scheme are used for the solid mechanics equilibrium and various solutions are possible for the pore pressure diffusion process. Consistency would imply to use fully explicit schemes for the two problems.

Moreover, it has been shown that time oscillations of the pore pressure may occur for other time schemes. Associated to the Terzaghi's postulate, oscillations could appear also on the stress tensor, what can degrade the numerical convergence rate for elastoplastic or elastoviscoplastic constitutive laws.

Large strains and large displacements have been analysed for solid mechanics. When solid mechanics is coupled with pore pressure diffusion, the Darcy's fluid velocity and the balance equations have to be computed in the geometry of the current configuration, which is changing from one iteration to the other.

Therefore a geometric coupling term appears in the iteration matrix when derivating the nodal water fluxes with respect to the nodal displacements. On the other hand, the solid and fluid specific weights have to be actualised taking into account the large strain process (Barnichon 1998).

When using isoparametric finite elements, the shape function for geometry and for pore pressure are identical. Let us consider for example a second order finite element. As the displacement field is of second order, the strain rate field is linear. For an elastic material, the effective stress tensor rate is then also linear. However the pore pressure field is quadratic.

Then the Terzaghi's postulate mixes linear and quadratic field, which is not highly consistent. Some authors have then proposed to mix in one element quadratic shape functions for the geometry and linear shape functions for pore pressure. But then problems arrive with the large strain geometry evolution and with the choice of spatial integration points (1 or 4 Gauss points?).

Numerical locking problems may also appear for isoparametric finite element when the two phases material (water + soil) is quite incompressible, i.e. for very short time steps with respect with the fluid diffusion time scale. Specific elements have to be developed for such problems .

**2.2.2. *Two fluids flow in rigid porous media coupling***

Flow in partly saturated rigid media is here considered. For unsaturated soils, the fluids are water and air. Often, the air phase is considered to be at constant pressure, what is generally a relevant approximation as air doesn’t affect highly the water flow. Then only one dof per node is sufficient, and the classical diffusion equations are relevant, with parameters depending on the suction or saturation level.

Flow in oil or gas reservoirs two or three fluids among oil, gas, condensates and water. Partial solving or mixture between different fluid are sometimes possible. Then two or more dof per node are to be considered.

The permeability and storage equation of each phase are depending on the suction or saturation level, and so the problem may be highly non-linear. However, coupling is not difficult to numerically be developed, as the formulation are similar for each phase.

**2.2.3. *Diffusion and transport coupling***

Heat and one fluid flow in a rigid porous media or salted water flow in coastal aquifers are concerned here. The fluid specific weight and viscosity is depending on the temperature or salt concentration, and the heat or salt transport by advection – diffusion process is depending on the fluid flow. Then a diffusion process and an advection – diffusion process have to be solved simultaneously.

Number of doff per node: 2 (fluid pore pressure and salt concentration of temperature).

**2.2.4. *Thermo-hydro-mechanical coupling***

The phenomena considered here (as for example for problems related to underground storage of nuclear waste disposals – (Gens 2001)) are much more complex as they associates multiphase fluid flow, hydromechanical coupling and temperature effects. All the features described in the preceding section are to be considered here, associated to some new points.

Heat diffusion has to be modelled. Temperature variation affects fluid flow, by a modification of the fluid specific weight or viscosity.

Moreover, if the two fluids concerned are a liquid and a gas (e.g. water and air), then equilibrium between the phases has to be modelled: dry air – vapour equilibrium.

Heat transfer is governed not only by conduction but also by advection by the liquid and gas movements. Similarly transfers of vapour and dry air in the gas phase are governed by diffusion and gradient of species density, but also but advection by the global gas movements. If the concerned geomaterials has a very low permeability (like clay for engineered barriers), then the diffusion effects will predominate and advection doesn’t necessitate specific formulation (Collin et all, 1999)

Finally the total number of dof per node is 5 for a 2D problem : 2 displacements, 2 fluid pore pressures and the temperature.

**2.3. *Finite element modelling: staggered approach***

Monolithical approach of coupled phenomena implies identical space and time meshes for each phenomenon. This is not always possible, for various reasons. The coupled problems may have different numerical convergence properties, generally associated to different physical scales or non-linearities. For example, a coupled hydromechanical problem may need large time steps for the fluid diffusion problem, in order to allow in each step fluid diffusion along distance of the order of magnitude of the finite elements. In the same time, strong non-linearities may occur in solid mechanics behaviour (strong elastoplasticity changes, interface behaviour, strain localisation…) and then the numerical convergence needs short time – loading steps, which should be adapted automatically to the rate of convergence. Then it is quite impossible to obtain numerical convergence for identical time and space meshes.

Research teams of different physical and numerical culture have progressively developed different problems modelling. As an example, fluid flow has been largely developed using the finite difference method for hydrogeology problems including pollutant transport, and for oil reservoir engineering taking multiphase fluid flow (oil, gas, condensate, water,…) into account. Coupling such fluid flow with geomechanics in a monolithical approach would imply to implement all the physical features already developed respectively in finite elements and finite differences codes. The global human effort would be very important.

Coupled problems are generally presenting a higher non-linearity level then uncoupled ones. Then inaccuracy in parameters or in the problem idealisation may cause degradations of the convergence performance.

How can we solve such problems and obtain a convincing solution? First of all, a good strategy would be to start with the uncoupled modelling of the leading process, and to try to obtain a first not too bad solution. Then one can add a first level of coupling and complexity, followed by a second one… until the full solution is obtained.

However such trick is not always sufficient. Staggered approaches may then give an interesting solution. In a staggered scheme, the different problems to be coupled are solved separately, with (depending on the cases) different space or time mesh, or different numerical codes. However, the coupling is ensured thank to transfer of information between the separated models at regular meeting points. This concept is summarised on the figure 3. It allows theoretically coupling any models together.

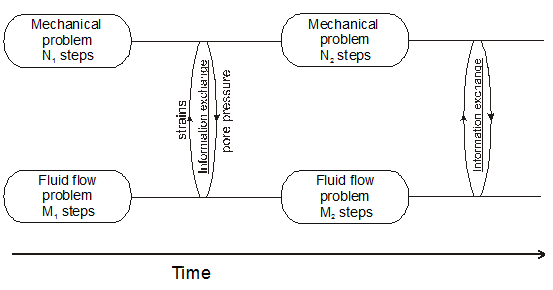


Figure 3. Scheme of a staggered coupling

When using different spatial meshes, or when coupling finite elements and finite differences codes, the transfer of information needs often an interpolation procedure, as the information to be exchanged are not defined at the points in the different meshes.

The accuracy of the coupling scheme will mainly depend on the information exchanges frequency (which is limited by the lower time step that can be used) and by the type of information exchanged. The stability and accuracy of the process has been checked by different authors (Turska et all 1993, Zienckiewicz et all, 1988). It has been shown that a good choice of the information exchange may improve highly the procedure efficiency.

**Questions for self-control**

1. What is the monolithic approach for finite element modeling?

2. What are the various terms of coupling?

3. What is the thermo-hydro-mechanical coupling

4. What is the staggered approach for finite element modeling?

5. How is the algorithm of the staggered approach for finite element modeling?

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**Lecture 5**

**Lecture topic: Review of Existing Models.** **Geertsma and Klerk’s Model. Perkins and Kern’s Model**

**The plan**

**1. Objective and approach**

1.1. Introduction

1.2. Objective

1.3. Approach

**2. Review of Existing Models**

2.1. Geertsma and Klerk’s Model

2.2. Perkins and Kern’s Model

**1. Objective and approach**

**1.1. *Introduction***

Hydraulic fracturing has been used to enhance well performance in applications ranging from oil production at great depths to environmental remediation at shallow depths. The process of hydraulic fracturing involves injecting fluid into carrying granular material, or the proppant into underground formations. The injection pressure increases until a critical value is achieved, and a fracture is initiated. The fracture will propagate away from the injection well and dilate as the injection continues. After the injection stopped, the proppant carried by fluid will remain in the subsurface and form a layer.

The forms of the hydraulic fracture differ when the state of stress, material property, and well bore perforation are different. In general, hydraulic fractures will propagate perpendicular to the least compressive stress. Hydraulic fractures created deep underground tend to propagate vertically, because the greatest compressive stress is vertical. Hydraulic fracture created at shallow depths can range from nearly horizontal to nearly vertical, because the state of stress at shallow depths is variable. Relatively high horizontal compressive stresses are common at shallow depths. Areas with this stress state are believed to favor the creation of flat-lying fractures. The maximum compressive stress is vertical in other areas, particularly those regions containing loosely consolidated material, and those areas are believed to favor the creation of steeply dipping to vertical hydraulic fractures.

Field tests were conducted by creating hydraulic fractures in overconsolidated silty clay glacial drift at a depth of 2 m [Murdoch, 1995]. Excavations revealed that these fractures are flat lying to gently dipping, disk-shaped structures [Murdoch, 1995]. Fractures initiated from a well bore will tend to propagate away from borehole. In addition, the mechanical process between *in situ* stress and free surface will cause the fracture to curve toward the ground surface [Pollard and Holzhausen, 1979]. The loading condition at the ground surface will affect the propagation pattern of the fracture. The fracture tends to propagate away from fracturing equipment located next to the borehole where the fracture is initiated [Murdoch, 1995].

Hydraulic fractures have been created in many different types of soils and site conditions, and results of more than 100 fractures are summarized in Murdoch [1995] and Murdoch and Slack [2002]. Recently more than 10 hydraulic fractures were created in silty clay derived from saprolite at a depth of 1.5 meters (5 ft). These hydraulic fracturing tests were conducted by Jim Richardson at Clemson University [Richardson, 2003]. Excavation of these hydraulic fractures showed a significant downward propagation in the vicinity of the borehole where the fractures are initiated. However, these downward propagations could happen as far as 1.5 m away from the injection well [Richardson, 2003]. The fractures showed a gently dipping, upward-curving trajectory after the downward propagation happened. As a result, it is clear that variations in the idealized form of shallow hydraulic fractures exist, and some of these variations could be important to the application of the hydraulic fracture.

Predicting the forms of hydraulic fractures has attracted a great deal of interest because of the widespread use of the technique by the petroleum industry. Hydraulic fractures created in oil reservoirs are typically vertical and they propagate roughly horizontally between an upper and a lower confining bed. This geometry, or a similar one, is assumed for nearly all of the models developed for applications in the petroleum industry. Although some of those models are extremely sophisticated, the geometry of fracture that they analyze differs markedly from that of fractures at shallow depths. Most of the shallow fractures that are useful are roughly flat lying. Perhaps just as important, shallow fractures can mechanically interact with the ground surface, and this interaction is believed to affect their form [Murdoch, 1995]. In a review did prior to this work, none of the hydraulic fracture simulators developed in the petroleum industry are capable of predicting the effects of a stress-free surface, the ground surface, because those effects are negligible at the depths of interest by the petroleum industry.

Some aspects of flat-lying hydraulic fractures can be predicted using analytical solutions described by Murdoch [2002]. However, that analysis assumes that the fracture form resembles a flat-lying disk. As a result, that analysis is unable to actually predict the form of a hydraulic fracture created at shallow depths, as reviewed by the author, a code capable of predicting those forms is unavailable.

**1.2*. Objective***

The primary objectives of this work are to develop a code to simulate the propagation of curving hydraulic fractures at shallow depths in two-dimensions, and to compare the results to existing analytical solutions and field observations.

**1.3. *Approach***

The approach to meet the objective of this study is to develop a two-dimensional hydraulic fracture simulator that couples FRANC2D, which is used to calculate stresses and displacements, with routines to determine fluid flow in the fracture and to maintain a propagation criteria. To be convenient, the newly developed FRANC2D coupled with fluid flow will be called as HFRANC2D in this work.

To test the new HFRANC2D, two models are established to simulate different settings, from a simple edge crack in elastic homogeneous media, to more complicated models intended to represent field conditions. The forms of hydraulic fractures in these numerical simulations, mainly described by driving pressure, aperture, and length are obtained from the simulator, and compared to the analytical solutions that are generally accepted [Tada, 1985; Geerstma and de Klerk, 1971; Murdoch, 1993; Murdoch, 2002].

**2. Review of Existing Models**

Hydraulic fracturing has been used in the oil industry since the 1940’s [Clark, 1949], and models of hydraulic fractures were published soon thereafter. Early understanding of hydraulic fractures is based on models developed by Geertsma and de Klerk [1969], and Perkins and Kern [1961]. Other analytical models were developed later and some of them are generally applicable to the work here because they can serve to verify the numerical solutions. Existing models relevant to this work are reviewed below. Many numerical models simulating vertical hydraulic fractures that propagate horizontally have been described by researches in the oil industry. That form is rare at shallow depths, so those models will not be reviewed here.

**2.1. *Geertsma and de Klerk’s Model***

Geertsma and de Klerk have investigated two fracture geometries; one is a vertical rectangular fracture with a fixed height that grows horizontally, and the other is a horizontal, circular fracture that grows radially [Geertsma and de Klerk, 1969]. The aperture of the vertical fracture is assumed to be uniform with fracture height (Figure 1.3-1), and the aperture of the radial fracture is assumed to be uniform around the circumference. The formation where fracturing occurs is homogeneous and isotropic and deformation is linear elastic. Fluid flow is laminar in the fracture. Gravity is not considered in this solution. Under condition of laminar flow, fluid flow in the fracture of a narrow aperture is under a resistance due to its viscosity. This resistance is balanced by decreasing the fluid pressure with the increasing of distance from injection point. The fluid pressure, aperture, and length can be used to describe the form of the fracture.

Fluid pressure is a function of fracture length, and the fracture length is a function of injecting time. So, the fluid pressure can be expressed as a function of time. The driving pressure , which is a result of subtracting injection pressure from *in situ* normal stress that is perpendicular to the fracture plane, can be directly expressed as [Geertsma, 1989]



, (1)

,

where:

 is the shear modulus,

 is the fluid viscosity,

 is the Poisson’s ratio.

Fracture length, , is the distance from the mouth of the fracture where it is initiated to the leading edge. It is decided by injecting time, injection system, and formation properties. Fracture length can be expressed in the form of a function of injecting time. Fracture aperture,, is the distance between two fracture faces. The aperture at the propagation starting point, for a fracture initiated by a well, also called fracture mouth opening, is of interest. The aperture can be related to injecting time by a power function [Geertsma, 1989].

, (2)

,

, (3)

,

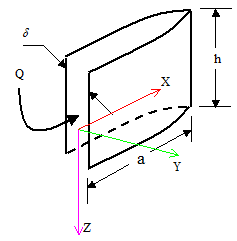


where

 is the fracture height, and

is the injection rate at the onset of the fracture.





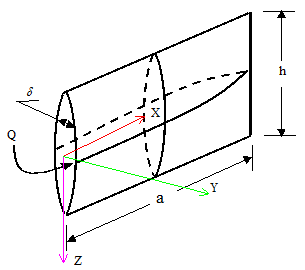


Figure 1.3-1 Vertical hydraulic fractures. Top one has rectangular cross section, bottom one has elliptical cross section. Both will only develop to planar fractures. [Geertsma, 1989]

A radially propagated hydraulic fracture is also modeled by Geertsma and de Klerck. The same assumptions are used for this kind of hydraulic fractures. The fracture has the same aperture at the same radial distance from the center of the fracture (Figure 1.3-2). The radius of the fracture is also referred to as the length of the fracture in the following discussion. The fluid that drives the fracture opening and propagating is injected into the fracture from the center of the fracture.

The driving pressure, fracture length, and fracture aperture at center with the changing of time could be expressed as follows [Geertsma, 1989]

, (4)

,

, (5)

,

, (6)

.

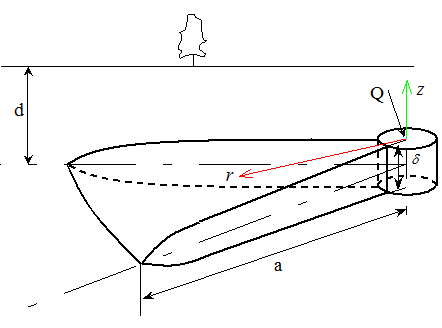


Figure 1.3-2 Horizontal hydraulic fracture. This fracture is created in deep rock, and will propagate radially [Geertsma, 1989].

**2.2. *Perkins and Kern’s Model***

A vertical hydraulic fracture with a fixed height, , which is a constant independent with fracture length is assumed. At each vertical cross section, it appears as an elliptic shape with the maximum aperture in the center. The hydraulic fracture is assumed to occur in a brittle elastic material under compression in their research. Under static conditions, the injection of fluid into a fracture can bring energy needed to propagate the fracture. But before the propagation happens, the fracture will only be inflated.

The driving pressure, , which is needed to propagate the fracture is [Geertsma, 1989]

, (7)

,

where

 is a constant decided by material elasticity, and injection rate.

This function showed that the fluid pressure needed to propagate a hydraulic fracture is an index power function of .

The aperture  at the injection point where the well is constructed can be related to injection time as a function in a form of [Geersma, 1989]

, (8)

,

where

 is a constant decided by elastic modulus, Poisson’s ratio, and injection rate.

So the aperture of the fracture at the opening will increase with time at a power function of .

The length of the fracture, which is noted as radius if the fracture is a radial fracture, has a relation with injection time as follows [Geertsma, 1989]:

, (9)

,

where

 is a constant decided by elastic modulus, Poisson’s ration, and injection rate.

In this analysis, the length of the fracture increases with injection time as a power function of .

The hydraulic fracture initiated horizontally and axial symmetric is another kind of case to have been studied. Under this situation, the fracture aperture may result from the compression of the surrounding material when the fracture is deep in the earth, or from the uplift of the overburden when it is in shallow depth [Perkins and Kern, 1961]. The cross-section of the fracture has an elliptical shape (figure1.3-2). The fluid flow in the fracture is assumed to be laminar low.

When the fracture is under deep depth of the earth, the driving pressure, fracture length, and fracture aperture at the center of the fracture (equations 1-10, 1-11, 1-12) has the similar power function of injecting time with equations (1-4), (1-5) and (1-6) [Perkins and Kern, 1961].

, (10)

,

 (11)

,

 (12)

.

When the fracture is in a shallow depth, if the radius of the fracture is large compared to the depth , then the fracture will possibly lift the overburden as well as compress the material under the fracture [Perkins and Kern, 1961]. This conceptual model will be similar with the analytical model that will be discussed later [Murdoch, 2002]. This solution may be used when , where  is the radius of the fracture,  is the depth from ground surface to the fracture [Murdoch, 2002].

**Questions for self-control**

1. What existing models do you know?

2. What is Geertsma and Klerk’s model?

3. What is Perkins and Kern’s model?

4. What are the main differences between Geertsma and Klerk’s model and the Perkins and Kern’s model?

5. How are these models determined?

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**Lecture 6**

**Lecture topic: Murdoch’s Analytical Model. Hydraulic Fracture Simulation**

**The plan**

**1. Murdoch’s Analytical Model**

**2. Tada Analytical Solution**

2.1. Summary of Analytical Models

2.2. Nilson’s Numerical Analysis

**3. Hydraulic Fracture Simulation**

**1. Murdoch’s Analytical Model**

The researches of hydraulic fractures in shallow soil were focused on onset of the hydraulic propagation. These works were lack of the capability to describe the forms of hydraulic fracture in shallow soil and its potential application in environmental remediation [Murdoch, 1993c]. Linear elastic fracture mechanics (LEFM) is used to perform the analysis since this has been widely applied to hydraulic fractures in rock [Geerstma and de Klerk, 1969; Perkins and Kern, 1971]. However, plastic deformation occurs in the silty clay, especially in the vicinity of the fracture tip [Murdoch, 1993c]. Lab experiments have showed that the size of the plastic zone near the tip of a hydraulic fracture created in soil is of the order of several millimeter comparing to the fracture length of nearly 10 cm [Murdoch, 1993c]. So the zone of plastic deformation could be ignored since it is small enough compare to the fracture length. After all, the LEFM is capable to be used to analyze fractures created in remolded silty clay in lab experimental conditions. The critical stress intensity  is found to have the capability to determine the onset of fracture propagation in silty clay during lab experiments [Murdoch, 1993c].

A conceptual model was built to describe the hydraulic fracture created in remolded silty clay during lab experiments. This is a two-dimensional geometrical feature with the length and aperture of the fracture changing during propagation, and the height of the fracture is roughly fixed [Murdoch, 1993c]. Plane strain condition at the leading edge of the fracture is assumed. During fracturing, two kinds of fluids were filled into fracture: one injected from the mid-line onset of the fracture and pore water infiltrated into the tip of fracture [Murdoch, 1993c]. The driving pressure induced by the injected fluid was assumed to be uniform since the flow rate is low so that the pressure gradient is small during the lab experiments. The magnitude of driving pressure is constrained by the propagation criterion that  through fracturing [Nilson, 1986]. Solutions for fracture length, driving pressure and aperture as a function of time are obtained after superposition of two loadings, one by injecting fluid, and another by infiltrated pore fluid [Murdoch, 1993c].

Hydraulic fractures are created in shallow fine-grained soil to improve environmental remediation processes that depend on subsurface flow [Murdoch, 1995]. The hydraulic fracture under such condition is a gently dipping feature that is slightly asymmetric with its initiation borehole (figure 1.3-3) [Murdoch and Slack, 2002]. An analytical solution was developed recently to describe the important features of such a hydraulic fracture [Murdoch, 2002]. The analysis was made based on the previous works discussed before [Nilson, 1986; Murdoch, 1993]. LEFM and critical mode I stress intensity  will be employed to predicting the details of propagation of hydraulic fracture.

The conceptual model of a hydraulic fracture at shallow depth can be condensed to a horizontal, circular, disk-shaped cavity loaded by internal fluid pressure and embedded in an elastic medium (figure 1.3-3) [Murdoch, 2002]. The soil material over the fracture cavity is treated as a flat-lying, thin elastic plate with certain elasticity features, since this fracture is produced in shallow soil. A driving pressure, which is a result of the subtraction of fluid pressure and overburden soil pressure, applied all over a circular area of the fracture. The radius of the circle is the length of the fracture since the fracture is axial symmetric.

By introducing a mass balance between injection and the fracture, and ignoring leak off, the analysis can be expressed as power functions of time. Thus, the driving pressure, radial length, and aperture at fracture center as function of time are derived as the following style [Murdoch, 2002]:

, (1)

,

, (2)

,

, (3)

,

,

where

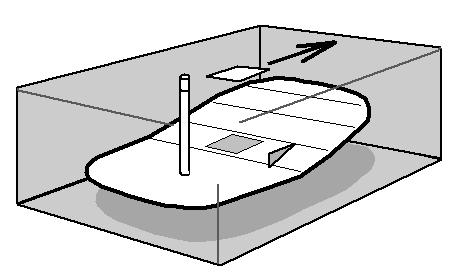
, , and  are the driving pressure, radial length, and aperture at the fracture center, respectively;

, , and  are three different constants decided by injection rate ,

soil elastic modulus  and Poisson’s ratio ,

critical mode I stress intensity ,

and the thickness of overlapped soil [Murdoch, 2002].



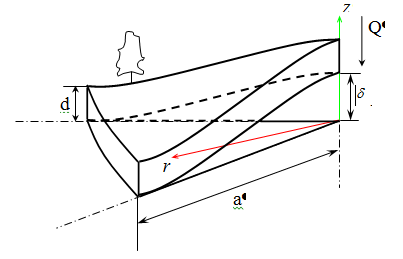


Figure 1.3-3 Horizontal hydraulic fractures in shallow depth. The generalized form of a hydraulic fracture (top) [Murdoch and Slack, 2002], and a cross section of a part of hydraulic fracture in shallow depth (bottom)

**2. Tada Analytical Solution**

The hydraulic fracture initiated from a well bore that opens at shallow depth is idealized to an axial symmetric, elliptical cavity, with uniform pressure applied on its surface [Tada et al., 1985]. This cavity lies in a linear elastic, infinite medium (Figure 1.3-4). The hydraulic fracture will be initialized and extended in the plane normal to the z-axis. Fracture aperture, which is the distance between two faces of the cavity, is measured along  axis. Since axial symmetry is assumed for  axis, only one direction of  axis will be studied. Accordingly, fracture length is defined as the distance starting form the center of the cavity to the edge of it (figure 1.3-4). This will reduce the conceptual model to a one-dimensional problem.

From the solution provided by Tada et al. (1985), The mode I stress intensity, , introduced by uniform loading in the fracture can be expressed as follows:

, (4)

where

 is the uniform pressure applied on the fracture surface,

 is the length of fracture, which starts from the center of the fracture to the leading edge of the fracture.

The aperture at the center of the fracture is [Tada et al., 1985]

, (5)

where

 is Poisson’s ratio,

 is Young’s modulus, and the volume of the fracture is [Tada et al., 1985]

. (6)

Assume the fracture is filled with an incompressible fluid with a constant injection rate , and then the time needed to fill the fracture volume is

. (7)

So the volume of the fracture can also be written in

. (8)

From (4) we have:

. (9)

Substitute (9) into (8), we then have:

. (10)

Figure 1.3-3 Horizontal hydraulic fractures in shallow depth. The generalized form of a hydraulic fracture (top) [Murdoch and Slack, 2002], and a cross section of a part of hydraulic fracture in shallow depth (bottom)

Then we get pressure as a function of time:

, (11)

.

From (8) we have

. (12)

Solve (12), (11) and (5), to get half-length of crack, , and aperture at crack mouth opening, , as functions of time, 

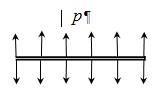
, (13)

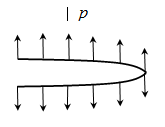
.

And

, (14)

.





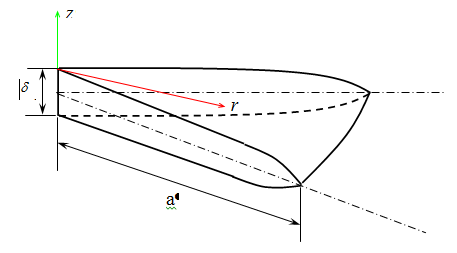
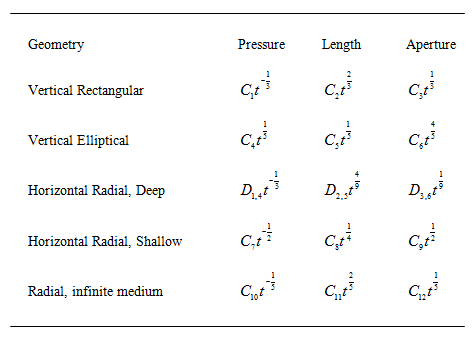


Figure 1.3-4 Tada’s analytical model. From top to bottom: initial fracture under uniform pressure, deformed fracture (center), 3-D view of the fracture (bottom).

**2.1. *Summary of Analytical Models***

The analytical models discussed in the above sections can all be expressed as simple power functions of injecting time (Table 1). According to the propagation direction, they can be divided into vertical and horizontal fractures. Vertical hydraulic fractures can have rectangular or elliptical cross-sections. Horizontal fractures may have elliptical or dome-like cross-sections.

**Table 1.** Summary of Analytical Models for Hydraulic Fractures



**2.2. *Nilson’s Numerical Analysis***

A hydraulic fracture with wedge-shaped cross-section and constant width can represent many hydraulic fracturing in the field [Nilson, 1988]. A general numerical approach for modeling hydraulic fractures is developed under a straining coordinate system, which elongates as the fracture propagates (figure 1.3-5). The material where the fracture initiates and propagates is linear-elastic. This elastic displacement is assumed to be quasi-static. The fluid flow that drives the fracture to propagate is injected form the opening of the wedge. However, the fluid flow is transient and compressible. A critical value of mode I stress intensity factor is used as the fracture propagation criteria. Fluid flow and elastic-solid coupling is directly treated by a control volume equation that governs fluid flow near the tip. To accomplish the coupling, the fluid pressure is treated as the primary dependent variable, and the aperture is built as a function of the fluid pressure and fracture length. The fracture is discretized into finite difference grids, where geometry and fluid flow calculation will be performed.

The computational procedure of Nilson’s analysis is employed by the following calculation procedures:

Using the current values of fluid pressure and fracture length, calculate the fracture aperture from elasticity.

Calculate the fluid flow from current pressures.

Calculate apertures at each finite difference grid.

Call a linear equation solver to solve pressure and length of fracture from aperture [Nilson and Griffiths, 1983].

The algorithm described above to simulate hydraulic fracture propagation coupled with fluid flow has been implemented, and compared to known analytical similarity solutions. The numerical results have shown convergence with these analytical results [Nilson and Griffiths, 1983]. This is a fast algorithm to simulate hydraulic fractures while keeps accuracy.

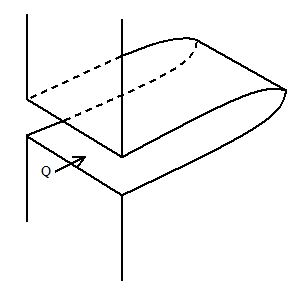


Figure 1.3-5 Fracture model for Nilson’s solution.

**3. Hydraulic Fracture Simulation**

Hydraulic fracture is a useful technique to enhance source recovery in known low permeability zone. However, The trajectory and shape of the fracture will decide the effect of the recovery in most situations. So, it is necessary and possible to simulate the behavior of the hydraulic fracture before fieldwork of utilizing fracture.

FRANC2D is one of the finite element fracture propagation simulators. The FRANC2D (Fracture Analysis Code) can perform two-dimensional fracture simulation based on linear elastic fracture mechanics (LEFM) [Wawrzynek and Ingraffea, 1996]. It is a highly interactive, windows-based program.

FRANC2D has been widely used in modeling fracture processes. FRANC2D has been used to simulate interacting fractures in rock [Lamkin et al, 1989], and to compute the stability, trajectory, and opening of two cracks observed near the toe of the Kolnbrein arch dam [Linsbauer et al, 1989]. Four models were built using FRANC2D to investigate the occurrence of cracks in the heel region of the dam [Linsbauer et al, 1989]. The application of this software to dam stability also includes a study of the Fontana dam [Ingraffea, 1990]. The fracture propagation trajectory in this dam was accurately reproduced by FRANC2D. FRANC2D has been used in a study to identify the differences in crack propagation between beam specimens and layered pavements. Two different crack growth rates are identified in pavements by this way, both due to the tensile stresses at bottom or top of the surface course [Castell et al, 2000]. FRANC2D is used to examine the effects of different interfaces on hydraulic fracture propagation [Carter et al, 1994]. Uniform crack pressure is applied to the crack face to simulate the near uniform fluid pressure when fluid flow is injected into the fracture. Hydraulic fractures initiation and propagation in uncased and cased wells are simulated. The simulation shows that a perfectly bonded casing and a casing with slipping interface both encourage propagation more than the uncased well [Carter et al, 1994].

The magnitude and variation of the stress intensity factor (SIF) has been studied by FRANC2D, when a problem of surface crack near Boussinesque load was studied [Alehossein and Korinets, 1997]. In another study, the fracture resistance properties of G550 sheet steels that are loaded in tension were studied. The mode I, crack opening, fracture toughness of G550 sheet steels is measured in lab. A numerical analysis of the stress intensity factor of cracks on structural performance in the elastic deformation range is completed using the FRANC2D finite element fracture simulator [Rogers and Hancock, 2001]. The simulation has revealed that the applied stress intensity factors at the time when the specimen failed has not reached the critical stress intensity factor. So the failure of the specimens may due to the yielding and ultimate rupture of the material, not the unstable fracture in the elastic deformation area [Rogers and Hancock, 2001].

FRANC2D has been modified to simulate crack growth in layered structures. The result is FRANC2D/L, which can model riveted and cohesive bonded structures such as lap joints, and to model tearing with the critical fracture tip opening angle approach [Swenson, and James, 1995]. FRANC2D/L also provided the capability to modeling fatigue crack propagation and studying the relation between residual strength and fracture length. FRANC2D/L is used to simulate the sleeve-fracturing process that is used to determine the far-field *in situ* stress [Charsley, Martin, and McCreath, 2003]. Pressure-displacement curves for sleeve fracturing under different boundary conditions were provided by using FRANC2D/L as an analysis tool. In aero structural engineering, FRANC2D/L is used to accomplish a damage tolerance analysis with the riveted lap joint design that will create complex loading conditions [Trego and Cope, 2001]. Analysis results of lap joint specimens showed good agreement between predicted and experimental lab results for a typical lap joint geometry. FRANC2D/L is used in this study to predict stress intensity factor in different stages of crack propagation. In a study of fatigue behavior of partially-penetrating K-welds in a cruciform joint, FRANC2D/L was used to calculate the opening mode and sliding mode stress intensity factor  and  [NyKanen, 1999]. A coupled finite element analysis and FRANC2D/L is used to study the fatigue cracking failure on high voltage stator winding end turns [Grise, 1997]. In this study, FRANC2D/L is used to investigate stress situations caused by temperature distribution, which is induced by high voltage (20000 Volts) electric current flow. Then a crack in this stress condition is propagated, and its path is investigated. This simulation has provided a perspective view of the impact of a fatigue crack to the stator.

However, with all the functionality that FRANC2D has, it cannot perform fracture propagation based on the comparison of the stress intensity factor and the critical value in an automatic pattern. Fluid flow cannot be simulated in the fracture propagation procedure. For a non-uniform pressure distribution, it cannot recalculate it and apply it along the extended fracture.

**Questions for self-control**

1. What is Murdoch’s analytical model?

2. What are the basics of Murdoch’s analytical model?

3. What is the Nilson numerical analysis algorithm?

4. How is hydraulic fracture simulation performed?

5. What is the Tada analytical solution?

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**Lecture 7**

**Lecture topic: Mechanical Analysis. Fluid Flow**

**The plan**

**1. Overview of analysis**

1.1. Approach

**2. Mechanical Analysis**

2.1. Fracture Tip Stress Functions

2.2. Determination of Stress Intensity Factor

2.3. Propagation Criterion

**3. Fluid Flow**

3.1. Analysis of fluid flow

3.2. Coupling Fluid Flow and Displacements Coupling

3.3. Fracture Inflation

3.4. Fracture Propagation

**1. Overview of analysis**

The hydraulic fracture propagation is analyzed by coupling mechanical and fluid flow analyses and by adopting a propagation criteria. Mechanical analysis will decide the propagation onset and direction based on stress condition in the fracture. The fluid flow analysis will decide the fluid pressure that will be driving pressure if no other loadings are applied in the fracture and its vicinity.

***1.1. Approach***

The propagation of a hydraulic fracture in soil is simplified to have two processes: the deformation of the medium where the fracture is initialized, and the fluid flow through the fracture [Murdoch, 1991]. The processes are coupled in that the fluid pressure resulting from fluid flow and leadoff to the formation is strongly dependent on fracture geometry, and the fracture geometry is on the other hand depending on the fluid pressure distribution. The deformation of the fracture geometry will be modeled using linear elasticity, and the fluid flow will be modeled using standard methods in fluid mechanics.

A criterion for fracturing is maintained throughout propagation. The propagation criterion requires the mode I stress intensity,  equal the critical stress intensity,  during the propagation. The fluid pressure will be adjusted so that this propagation criterion is satisfied at each fracture length.

The direction of the propagation of the hydraulic fracture is decided by the position of the maximum tensile stress in the vicinity of the crack tip, and this can be determined using the mode I, and mode II stress intensity factors, *KI* and *KII*.

**2. Mechanical Analysis**

The mechanical analysis of hydraulic fracturing is conducted based on Linear Elastic Fracture Mechanics (LEFM) to decide propagation criterion, linear elasticity to decide fracture geometry, and fluid mechanics.

***2.1. Fracture Tip Stress Functions***

A crack in a solid can be under three different stress conditions (figure 2.2-1). Normal stresses will create an “opening mode” or Mode I loading. The displacement of the crack surface is perpendicular to the plane of the crack. Shear loadings in the same plane, as the crack face will result in Mode II or “sliding mode”: the displacement of the crack surfaces is in the plane of the crack and perpendicular to the leading edge of the crack. Out-of-plane shear loading will cause the crack to exist in “tearing mode” or mode III. Under this circumstance, the crack surface displacements are in the plane of the crack and parallel to the leading edge of the crack.

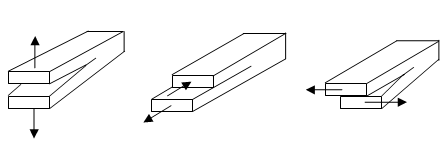


Figure 2.2-1 Three modes of fractures. Mode I, opening (left); mode II, sliding (middle); mode III, tearing (right).

For the mode I fracture, the stresses and displacements in the vicinity of the crack tip are given by Irwin as (1958) [Lawn, 1993]:

 (1a, b)

For mode II there are

 (2a, b)

where

 (3)

For plane stress, or

 (4)

For plane strain,  is Young’s modulus,  is Poisson’s ration.  is the displacement parallel to the  coordinate, and  is the displacement normal to the  coordinate.

The stress and displacement equations in (1a,b, 2a,b) can be simplified to simple forms,

 (5a)

 (5b)

In which the function of  is decided by spatial coordinates of fracture tip. The  factor depends only on the outer boundary conditions, i.e. loading condition

(), and specimen geometry (  ). For , which is at the tip of the fracture, a singularity exists in the linear stress and strain. The  factor becomes the strength at this point.

***2.2. Determination of Stress Intensity Factor***

To calculate mode I stress intensity in a finite element mesh (figure 2.2-2), take three points on the fracture, name the point at the tip as “A”, other near tip points with “B” and “C”. Under pure mode I condition, at a finite grid block A-B,

 (6)

and

 (7)

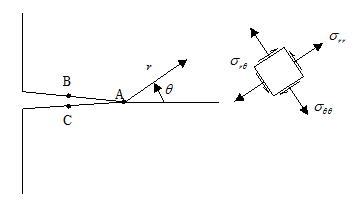


Figure 2.2-2 Stress field at fracture tip. The radial coordinate system is showing here. A, B, C are three point that are used in finite element calculation. A is the fracture tip location.

Since the fracture is under pure mode I loadings, the displacement parallel to the  coordinate is zero. Only the displacement in the  direction that is normal to the  coordinate exists. So from (1) we have the difference in displacements between point A and point B [Ingraffea, 1983]

 (8)

So the mode I stress intensity factor,  is computed in the finite element methods by

 (9)

For pure mode II condition, there is no displacement in the direction normal to the  coordinate, so  is zero. Also consider finite element grid A-B, the displacement parallel to the  coordinate is

 (10)

So the mode II stress intensity factor,  is computed in the finite element methods by

 (11)

Equations (8) and (10) are applicable for cracks with symmetrical structure, which means the displacements at point B and point C are identical in magnitude and opposite in sign. However, in a mixed mode I-II scenario occurs when a free surface overlies the fracture [Pollard and Holzhausen, 1979].

Under mixed mode I-II condition the displacements at two sides of the fracture face are different. In the case discussed above, the displacement normal to the fracture axis at points B and C,  and , respectively, are different; and the displacement parallel to the fracture axis at these two points,  and , are different too. At point C, the condition is

 (12)

And

 (13)

The normal displacement  at point B only results from mode I condition, so

 (14)

And the same situation occurs at point C, where

 (15)

where . Subtract (14) and (15), and solving for , gives

 (16)

Under mode II condition of the asymmetric mix-mode loading condition, the displacement normal to the fracture axis is zero at both points B and C. However, the displacement parallel to the fracture axis at point B is

 (17)

At point C is

 (18)

Solve (17) and (18) for , get

 (19)

(16) and (19) are the general forms for computing stress intensity factors using displacements method. This approach is called displacement correlation method, which is one of three possible methods used to calculate stress intensity factor in FRANC2D. In the displacement correlation method, displacement values computed at the finite element grid nodes near crack tip are correlated with analytical solutions [Ingraffea, 1983]. This is the default method that FRANC2D will use in its simulation.

***2.3. Propagation Criterion***

The critical value of the stress intensity factor, , which is also widely recognized as fracture toughness [Lawn, 1993], is a material property used to describe the resistance of a material to fracture propagation. The fracture toughness of a material is related to the strength of bonds between constituent particles and the size of flaws in the material. Fracture toughness is widely used to describe fracture propagation through rock, and it appears to be a valid predictor of fracture propagation in cohesive soils. In rock, fracture toughness values are typically on the order of 1 MPa m1/2, whereas the fracture toughness of partially saturated silty clay is less than 0.05 MPa m1/2 (Murdoch, 1993 b).



The propagation criterion that will be used in this work is that under mode I loading condition, the onset of the fracture propagation will appear, and the propagation will continue as far as the criterion for fracture advance

 (20)

is satisfied. The fracture will cease propagation when

 (21)

*Propagation Direction*

In a pure mode-I fracture loading condition, the fracture will propagate in the same direction as it was initiated. However, under mixed mode-I and mode-II fracture loadings, the initiation and propagation of a fracture is decided by not only the mode I stress intensity factor and critical value, but also mode II stress intensity factor and critical value.

One theory that has been used in FRANC2D to simulate mixed mode-I and mode-II fracture propagation is the maximum circumferential tensile stress, , near the crack tip [Ingraffea, 1987]. This theory has the following important ideas:

1. Crack extension starts at the crack tip in a radial direction.
2. Crack extension starts in a plane normal to the direction of greatest tensile stress.
3. Crack extension begins when  reaches a critical material-constant value.

The theory above can be implied by adding equation (1b) with (2b), and this will give:

 (22)

and

. (23)

From equation (23), the direction of the initial fracture increment, , can be found from

. (24)

For example, for pure mode I,

, (25)

. (26)

So the crack will propagate in its own plane. However, for pure mode II,

, (27)

. (28)

That is, under pure mode II the crack will tend to curve at an angle of 70.5 degree. Under mixed mode conditions, the fracture will propagate at an angle between  and , according to the mixed condition of  and .

In the numerical simulation of hydraulic fracture performed later by FRANC2D, the maximum circumferential tensile stress theory will be used to decide the propagation angle of the hydraulic fracture in shallow depth.

**3. Fluid Flow**

Hydraulic fractures are propagated using pressurized fluid as a driving force. The behavior of the fluid obeys the theory of fluid mechanics. One-dimensional laminar flow of a Newtonian fluid is assumed to occur in the fracture.

***3.1. Analysis of fluid flow***

Fluid flow in the deforming hydraulic fracture will be analyzed by integrating the continuity equation using finite difference methods. It is assumed that fluid flow occurs in one dimension along the long axis of a rectangular fracture or along the radial direction of an axial symmetric fracture. The fluid is assumed to be compressible, with the compressibility

,

where

 is the fluid density and

*P* is pressure.

Even though flow is one dimensional along the axis of the fracture, it can leak out through the fracture walls, and the flux through one wall of the fracture is *q*L. Taking a control volume on the fracture where the fracture walls form two of the boundaries of the control volume allows mass conservation to be written as

, (29)

where the term in square brackets is used for axial symmetric problems, but the term in square brackets is set to zero for linear problems. Assuming flow through the fracture is laminar, the flux or average Darcy velocity through the fracture is

, (30)

where

*h* is hydraulic head,  is unit weight of fluid and

 is the dynamic viscosity of the fluid.

The fracture is discretized into *n* finite difference elements of length, *r.* Eq. (30) is substituted into eq. (29), and the derivatives are expanded in finite difference form. This results in *n* equations that are solved simultaneously by writing in implicit form and using the Thomas algorithm [Wang and Anderson, 1982]. The implicit form of the finite difference equations other than the boundaries is

, (31a)

where the vectors are

, (31b)

, (31c)

, (31d)

. (31e)

And for convenience

 (32a)

 (32b)

 (32c)

 for axial symmetric model, or  for a linear model  (32c)

All values are taken to be at the current time step, m, except the terms with m-1 as a superscript and they are taken at the previous time step.

Boundary conditions during propagation are set to maintain a specified hydraulic head at the fracture tip, and a specified flow rate into the mouth of the fracture. Conditions at the mouth of the fracture are satisfied by requiring

, (33a)

, (33b)

, (33c)

, (33d)

where

*L* = *H*, the fracture height,

for a linear model and *L* = 2ƞrw forr an axial symmetric model.

The volumetric flow rate into the fracture is *Q*. The terms above in square parentheses are used for an axial symmetric model, but they are set to zero for a linear model. The tip is held at constant head, *h*tip, by requiring

 (34)

The system equations outlined above in eqs. (31) through (34) can be solved implicitly using the Thomas algorithm. The solution requires accessing values of *h* and at the previous time step in order to build the vector *m*.

***3.2. Coupling Fluid Flow and Displacements Coupling***

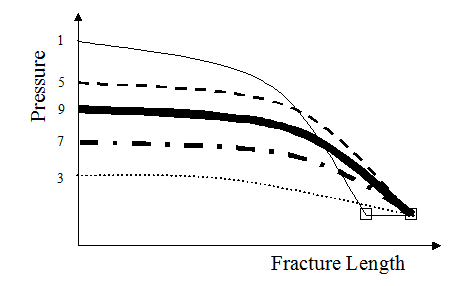
The analyses of fluid flow outlined above are coupled to the mechanical analyses of displacements through the fracture aperture. Changes in fluid pressure or hydraulic head result in changes in fracture aperture, and this in turn affects the hydraulic head because it enters the *m* vector specified above.

The coupling between mechanical displacements and fluid flow is performed iteratively (figure 2.3-1). The basic scheme during time stepping is to assume an array of apertures that are the same as the previous time step, and then solve for the hydraulic heads using the approach outlined above. This will produce an array of heads that overestimates the correct solution because the apertures that were used were too small. Then, the apertures are recalculated using the new values of head. This is done using FRANC2D. The heads are recalculated using the new trial values for aperture. The new array of heads will be less than the previous values. Apertures calculated using the new array of heads will be less than the previous values, but greater than the original ones. Alternating calculations of aperture and head in this manner will converge if suitably small time steps are taken. One method for improving the rate of convergence is to use a weighted average of the previous two aperture values, instead of simply the last value. The apertures oscillate between too large and too small, so taking the average tends to dampen this oscillation and improve convergence. A similar method is suggested by Nilson and Griffiths [1983].

This iteration process produces fractures that have different volumes whenever the aperture is recalculated. Generally, the assumption is that fluid is injected into the fracture at a constant rate. This means that the time step, *t*, changes with every iteration. This is accomplished by updating the time step in the routine where the heads are calculated. Convergence is evaluated using the updated time step.

The iteration procedure converges when consecutive hydraulic heads differ by less than some tolerance set by the user. A maximum relative error between successive iterations of 0.01 was used to indicate convergence for this work.

The analysis requires selecting the hydraulic head at the tip of the fracture. This head is selected using two methods, depending on whether the fracture is inflating or propagating.



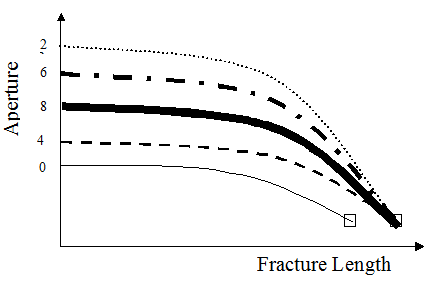


Figure 2.3-1 Coupling analysis of flow and fracture deformation. The number is the order that pressure and aperture are calculated. The figure is not to scale.

***3.3. Fracture Inflation***

The Inflation of an existing hydraulic fracture is analyzed as follows:

1. Initialize the fluid flow. Get the fracture length from mechanical analysis, use this length to define finite difference grid size.

2. Assume a trial uniform hydraulic head along the fracture. This head should produce a stress intensity factor that is less than the fracture toughness.

3. Determine trial aperture from the trial head just assigned using elasticity analyses in FRANC2D. Hydraulic head is used in the flow analyses, but head is converted to fluid pressure for the mechanical analyses.

4. Iterate between solutions of aperture and head until convergence.

5. Calculate the stress intensity factor using FRANC2D. Compare this value to the fracture toughness.

6. If the stress intensity is less than the toughness then the fracture is assumed to be inflating, but not propagating. The hydraulic head at the fracture tip is increased and the procedure is repeated and the fracture is inflated until the stress intensity equals the toughness

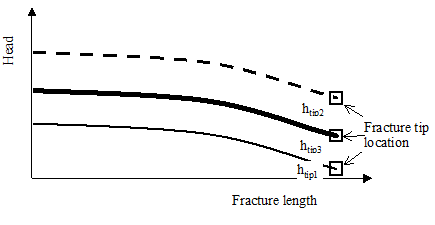
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This is the criterion for propagation.

***3.4. Fracture Propagation***

The fracture propagation analyzed by solving for the fluid flow as described above, but here the fracture length is increasing and the hydraulic head at the fracture tip is adjusted to maintain the propagation criterion. This is the sequence of steps that is used:

1. Increment the fracture length. The angular position of the new increment, , is determined using root of eq (24) (figure 2.3-2).
2. Calculate hydraulic heads using the analysis described above. The aperture during the previous time step, *h*m-1 is assumed to zero in regions that were unfractured in the previous time step. The hydraulic head at the fracture tip, *h*tip, is assumed to be equal to the head at the previous time step. Iterate as above until convergence.
3. Determine the stress intensity factor. This value will exceed the toughness because the fluid pressure is unchanged but the fracture is now longer (figure 2.3-2, htip2). Decrease head at the tip by a small amount, typically several percent of the current head at the tip, and recalculate the stress intensity, *K*I.
4. Use the pairs of values of *h*tip and *K*I to calculate a new value of *h*tip so that *K*I = *K*IC (figure 2.3-2, htip3). A linear interpolation or extrapolation scheme is used to estimate a new value for *h*tip.
5. Compare the new value of *K*I to *K*IC. If the two are within a tolerance value, then a solution has been obtained for the equilibrium head and aperture of the fracture at the current length. Otherwise, if *K*I differs from *K*IC by more than the tolerance then repeat the interpolation and try again. Typically, values of *K*I were required to be within 1 percent of *KIc*. This commonly occurs with only one interpolation.
6. Steps 1 through 6 are repeated until the fracture has reached a desired length. Mass balances are checked after each increment of length.



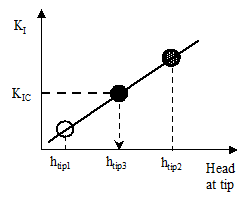


Figure 2.3-2 Fracture propagation analysis. The head at the tip is adjusted from Ptip1, to Ptip2, then Ptip3. Figure is not to scale.

**Questions for self-control**

1. How are fracture tip stress functions calculated?

2. How is the stress intensity factor determined?

3. What is the formulation of the propagation criterion?

4. What are the basics of fluid flow analysis?

5. How is the propagation criterion formulated?

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**Lecture 8**

**Lecture topic: Finite Element Modeling of Non-Viscous Incompressible Fluid Flow and Convective Temperature Flow**

**The plan**

**1. Governing Equation of Two Dimensional Flow**

1.1. Introduction

1.2. Problem Definition

1.3. Governing Equation of Two Dimensional Flow

1.3.1. Governing Equation for Fluid Flow Analysis

1.3.2. Governing Equation for Temperature Flow Analysis

**2. Finite Element Modeling**

**3. Flow Analysis**

3.1. Fluid Flow Analysis

3.2. Temperature Flow Analysis

3.3. Conclusions

**Appendix I**

**1. Governing Equation of Two Dimensional Flow**

**1.1. *Introduction***

Finite Element Method (FEM) is considered very powerful and efficient in solving partial differential equations. Exact solution of some engineering applications, such as fluid flow analysis, is a challenging task. However, FEM can be used to model these problems and give solution near to exact by using stream function approach. In this study, FEM is employed to analyze a non-viscous incompressible fluid flow inside a pipe and then solve for the heat flow transfer through the same pipe. The fluid flow is expressed by partial differential equation (Poisson’s equation). While, heat transfer is analyzed using the energy equation. The domain is discretized using 560 elements and that corresponds to a total number of nodes 616. A four-node isoparametric quadrilateral element is used to model the domain. The stiffness bandwidth is assured by using bandwidth reduction programs. A previously developed Matlab codes are used to perform the analysis. Results showed that both fluid flow and temperature flow are influenced significantly with changing entrance velocity. Also, there is an apparent effect on the temperature flow fields due the presence of an energy source in the middle of the domain.

Finite element method techniques are widely used in problems that require solution of partial differential equations. In this method of analysis, a complex region defining a continuum is discretized into simple geometric shapes called finite elements. The material properties and the governing relationships are considered over these elements and expressed in terms of unknown values at element corner. An assembly process duly considering the loading and constraints, results in a set of equations. Solution of these equations gives us the approximate behaviour of the continuum. Thompson [1] discusses solution of partial differential equations involved in areas such as Fluid Mechanics, Elasticity and Electromagnetic Field by using FEM. Details about fluid mechanics and heat transfer problems and their solution can be found in [2]. A more complex transient heat conduction equation is discussed in Winget and Hughes [3]. Similarly Johan *et al.*[4] and Jacob and Ebecken [5] develop step size selection schemes based on heuristic rules for compressible Navier-Stokes equations and structural dynamics problems respectively. Convective heat transfer or, simply, convection is the study of heat transport processes by the flow of fluids. Problems related to convective heat transfer rest on basic thermodynamics and fluid mechanics principles, which essentially involved with partial differential equations.

In this ltcture, a two-dimensional non-viscous incompressible steady flow problem is solved using the finite element method through solving partial differential equations of the fluid flow. The flow field of that fluid is then employed to solve the partial differential equations of temperature flow. For both fluid flow and temperature flow, boundary conditions are applied. The domain of the problem is discretized to a large number of elements to assure the accuracy of the solution. Four-node isoparametric quadrilateral elements are used to model this problem. The influence of the presence of heat source inside the domain on the temperature flow field is also studied. Analyses are done by studying both flow field and temperature field for various values of entrance velocity. Analyses show very important results, such as the temperature field in decreasing by increasing the entrance velocity.

**1.2. *Problem Definition***

A fluid passes through a domain represented by a pipe having dimensions of 20x50 units in the two-dimensional plan, as shown in Figure 1. The fluid flow enters from the left with a uniform velocity *V*, as illustrated. It flows around a small pipe located at the middle of the domain and having a diameter *d* = 7.5 units, as shown in Figure 1. This small pipe contains an energy source produces *q* units of energy per surface area of the pipe per unit time. As the fluid passes through the domain, the velocity of the fluid changes due to the presence of the small pipe at the middle. Also, due to convection phenomenon, heat energy is carried out by the fluid flow through the pipe domain. Thus, the temperature flow field changes by changing the velocity of the fluid flow.

By assuming that the fluid is non-viscous and incompressible, both velocity and temperature fields are independent of each other. As such, the governing partial differential equations that describe the fluid flow and the temperature flow can be solved independently. The fluid flow is solved using the stream function for a specific values of entrance velocities, as will be discussed later. Once the flow is determined, the temperature flow can be established by solving temperature governing equation.

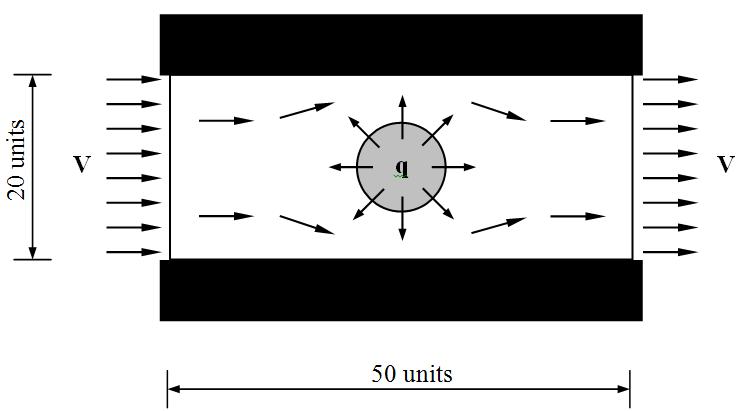


Figure 1. Steady Flow System Definition

**1.3. *Governing Equation of Two Dimensional Flow***

The governing equation that govern the two-dimensional problem, in general, can be obtained from the relation

 (1)

in which the coefficients , *G* and *H* are given as functions of x and y. The two-dimensional domain is denoted by . *S* is the boundary of the domain and *S1* and *S2* are parts of that boundary and satisfy .  and  represent the outward normal unit vectors on the boundary.

**1.3.1. *Governing Equation for Fluid Flow Analysis***

Poisson’s equation is originated from the general governing partial differential equation and it is adequate to describe a large number of applied problems including the flow of ideal fluids problems. The two governing equations for the flow of ideal fluids are

 (Irrotational flow) (2)

 (Incompressible flow) (3)

If we define  such that it identically satisfies the condition for incompressible flow, then we have

 (x component of the velocity) (4)

 (y component of velocity). (5)

By substituting equations (4) and (5) into equation (2), the stream function of the fluid flow can be written as follows

. (6)

**1.3.2. *Governing Equation for Temperature Flow Analysis***

For steady state two-dimensional convection through a constant-property homogenous fluid, the energy equation is given by (Reference 2)

. (7)

or in other form

. (8)

where

 is the temperature,

*k* is the thermal conductivity of the fluid and

is the heat capacity of the fluid.

 and  denote the velocity components in x and y directions, respectively.

Since the time-dependency is beyond the scope of this paper, the thermal conductivity of the fluid will be assumed to be time independent and take a constant value *k* = 1.0. Also, the heat capacity is assumed to be equal to 1.0.

**2. Finite Element Modeling**

The problem is discretized using a previously developed Matlab code mesh generator “mesh.m” provided in Reference [1]. The input data for the mesh generator are the number of generation loops and some geometric coordinates of specific points on each side of each loop. Also, the number of divisions per each side is required to specify the number of elements per each loop. In the current study, six loops are used to generate the mesh of the domain (Figure 2). Thus, the total used number of elements is 560 elements and that corresponds to a total number of nodes 616. A four-node isoparametric quadrilateral element is used to model the domain to maintain the continuity of the degrees of freedom along the edges of the elements. It should be noted, as shown in Figure 2, that the sizes of the elements decreases with getting closer to the small pipe at the middle to ensure the accuracy of the solution at this area of concentrated stresses. The mesh discretization provided in Figure 2 will be used in both the flow and temperature analyses.

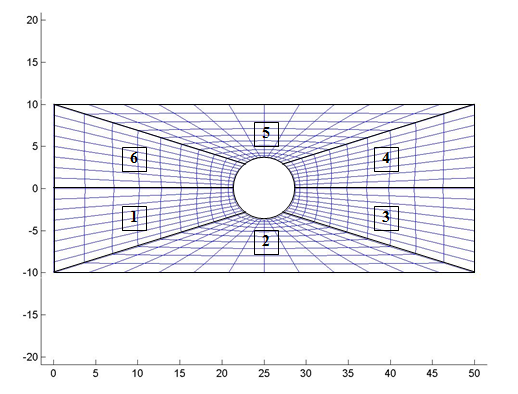


Figure 2. Mesh Discretization of the Domain (560 elements and 616 nodes)

A Matlab user INCLUDE code denoted by NPCODE.m (provided in Reference 1) is included in mesh.m code immediately before the output is saved. The purpose of this code is to assign NPcode values to each node. These values have meaning only to the user and are not used in any of the finite element codes. However, the user can use these numbers in any other supplement code. The NPcode values are often used in the INITIAL.m Matlab user INCLUDE code (will be discussed in later section) that set boundary and initial conditions for a particular finite element analysis. The output files of the mesh generator program “mesh.m” are ASCII files; MESHo, NODES and NP. The MESHo file contains the number of nodes, number of elements, and number of nodes per element, respectively. The NODES file includes three columns and number of rows equal to the total number of nodes. The first two columns contain X and Y coordinates of each node, while that last column contains the NPcode value for each node. Finally, NP file includes the connectivity array between the elements.

A problem encountered with the two-dimensional problem analyses is how to number the nodes such that it minimizes the storage needed for the stiffness matrix. The bandwidth of the stiffness matrix depends on the way the nodes have been numbered. Also, the difference between a good numbering scheme and a poor numbering scheme can result in a very large difference in bandwidth requirements. Since finite element equations are related to each other only through common elements, reduction of the bandwidth needs nodes that are connected by common elements be as close in numerical value as possible. A previously developed bandwidth reduction program developed using Matlab code denoted newnum.m (Provided in Reference 1) is used in the analysis to specify a new numbering scheme that is able to reduce the bandwidth of the stiffness matrix. The ASCII files that are resulted from the mesh.m program are used as input files for the newnum.m program.

The code described in the newnum.m program begins with the specification of one or more nodes as starting nodes. The user specifies these nodes be the first nodes in the new numbering scheme. For example, if five nodes are designated, the line connecting these nodes is referred to as the first wave of nodes (Figure 3). The second wave consists of all nodes that are linked to nodes in the first wave through common elements. The nodes in the second wave are then given the next consecutive numbers in the new order. This process continues until all nodes have been given new numbers. Finally, all elements should have node numbers that differ by no more than the number of nodes in the longest two consecutive waves, and even less. The output of newnum.m program is an ASCII file denoted NWLD, which contains the new numbering scheme that will be used in the finite element analysis.

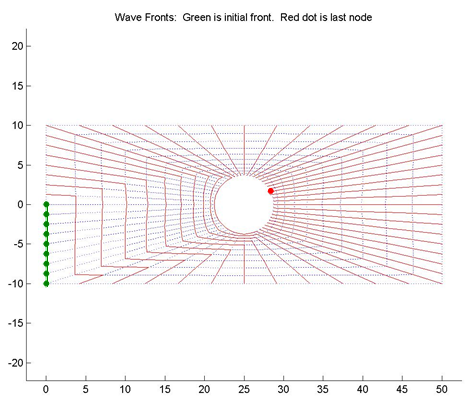


Figure 3. Wave Fronts for New Numbering Scheme (Green is first wave and red dot is last node)

**3. Flow Analysis**

Both fluid flow and temperature flow can be described by the general equation that governs the two-dimensional problems (Equation (1)). However the coefficients of the governing Equation (1) (, *G* and *H*) differ from the fluid flow to the temperature flow such that they satisfy the governing equations given in Equations (6) and (8) for fluid flow and temperature flow, respectively. The previously developed Matlab program denoted steady.m is used to conduct the finite element analyses for fluid and temperature flow. The flow chart that describes this program is given in Figure 4 and the source code of the program is enclosed in Appendix I.

**3.1. *Fluid Flow Analysis***

The fluid flow is characterized by the Poisson’s equation given by Equation (6) and recalled here for convenience

,

in which the coefficients of the general governing equation (1), , *G* and *H*, take the following values; 1.0,  1.0,  0.0, 0.0, *G* = 0.0 and *H* = 0.0. These coefficients are incorporated in the user INCLUDE Matlab program denoted COEF.m, which is called from inside the steady.m program to define these coefficients.

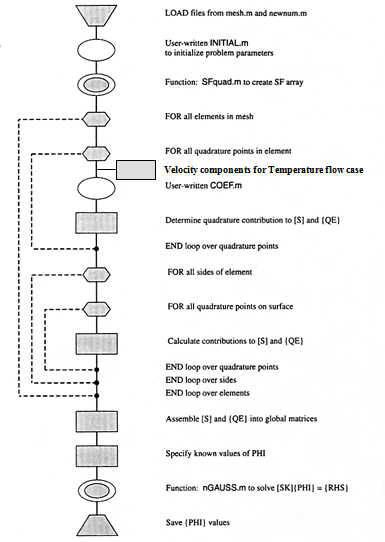


Figure 4. Flow Chart of steady.m Program

The boundary conditions for the fluid flow analysis are illustrated in Figure 5.

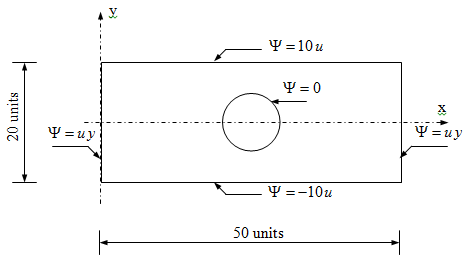


Figure 5. Boundary Conditions of the Fluid Flow

The x-component of the velocity at the entrance and at the exit is constant. As such, integrating equation (4) for the x-component of the velocity gives the fluid flow conditions at entrance and at exit as

. (9)

in which,  is constant specific values for the velocity and *y* is the y-coordinates of the nodes in that boundary. Since, the flow is non-viscous and incompressible, the x-component of the velocity maintains constant at the upper and lower boundaries. Thus, the boundary conditions of fluid flow at upper and lower boundaries can be given by the relations;

 (at upper boundary) . (10)

 (at lower boundary) . (11)

where the values 10 and –10 are the y-coordinates of the nodes on upper and lower boundaries, respectively. The y-component of the velocity, i.e.  , is constant and equal to zero the at entrance and the exit. Also, to maintain the non-viscosity and incompressibility of the fluid, the fluid flow at the small circular pipe at the middle of the domain is taken equal . These boundary conditions are specified in the INITIAL.m Matlab program code (enclosed in Appendix II), which is called from inside the steady.m program to define the boundary conditions of the problem. Once the boundary conditions and the coefficients are defined in the steady.m program, the analysis of the fluid flow is determined. Results are obtained for different values of the entrance velocity = 0.0, 0.30, 0.60 and 1.0. Figure 6 shows the flow field of the fluid at the different specified values of the entrance velocity. As illustrated from the results that the flow field is the same for all values of the velocities, however, the absolute values of the flow increases with increasing the entrance velocity. Also, it is noticed that when the velocity = 0.0, the flow field is static and the stream function is constant all over the domain.

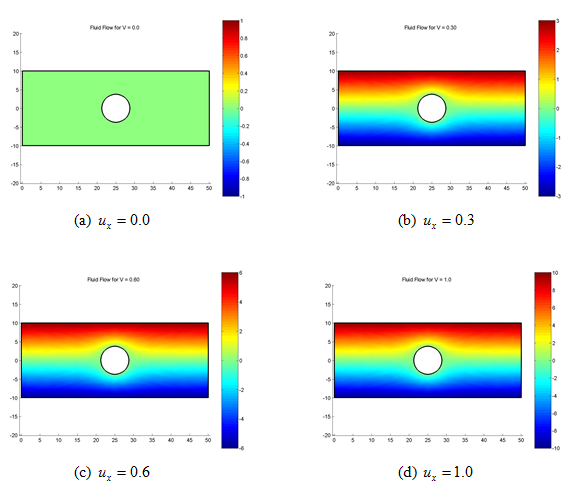


Figure 6. Fluid Flow Field for Different Values of Entrance Velocity

**3.2. *Temperature Flow Analysis***

The temperature flow is characterized by the energy equation given by Equation (8) and recalled here for convenience

,

in which the coefficients of the general governing equation (1), , *G* and *H*, take the following values; , ,  , , *G*= 0.0 and *H* = 0.0. The thermal conductivity  and the heat capacity of the fluid are assumed to have unit values. The coefficients  and are functions of velocity components  and , respectively. Recalling that the velocity components  and  are given by using equations (4) and (5) as derivatives of the fluid flow with respect to x and y, respectively. As such, a small Matlab code is incorporated in the steady.m program at the position shown in the flow chart (Figure 4) just before calling the user’s INCLUDE file COEF.m to calculate the velocity components  and . This Matlab code is illustrated as follows;

% -----------------------------------------

% Calculating velocity components

% -----------------------------------------

for K=1:NNPE;

NPK=NP(I,K);

VP060(K,I)=VP06(NPK);

ux1= ux1+DNDY(K)\*VP060(K,I);

uy1= uy1-DNDX(K)\*VP060(K,I);

end

in which NNPE is the number of nodes per element, NPK is the global numbering of each node, NP is the array of global numbering of each nodes as function of element number I and local node numbering K. The derivatives of the flow functions with respect to x and y at each node are denoted by DNDY(K) and DNDX(K), respectively. VP06(NPK) is the fluid flow function values, , resulting from fluid flow analysis for entrance velocity equal = 0.6 given for the global numbering of the nodes. VP060(K,I) is the two dimensional array of the fluid flow function  given at each local node per each element. Once the velocity components  and  are calculated, the steady.m program call for the user’s INCLUDE code COEF.m to read the rest of the coefficient as;

, ,  , , *G* = 0.0 and *H* = 0.

The boundary conditions for the temperature flow analysis are illustrated in Figure 7. The small pipe at the middle is assumed to produce *q* units of energy per surface area of the pipe per unit time. The energy q is considered equal to unity at the small pipe and equal to zero at exit, lower boundary and upper boundary. The temperature flow  is considered equal to zero at entrance. These boundary conditions are employed in the INITIAL.m Matlab program code (enclosed in Appendix III), which is called from inside the steady.m program to define the boundary conditions of the problem. Once the boundary conditions and the coefficients are defined in the steady.m program, the analysis of the temperature flow is ascertained.

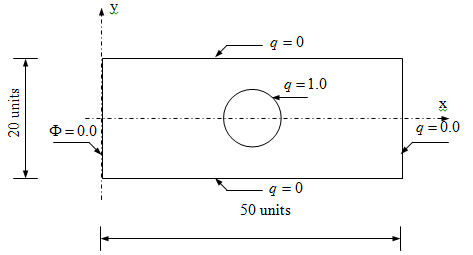


Figure 7. Boundary Conditions of the Temperature Flow

Results are obtained for different values of the entrance velocity = 0.0, 0.30, 0.60 and 1.0. Figure 8 shows the flow field of the temperature at the different specified values of the entrance velocity. It is noticed from the results that there is a steep decrease in the temperature field with increasing the entrance velocity. This coincides with the fact that heat transferred by convection is greater than heat transferred by conduction. Also, Figure 8 illustrates that the rate of heat transferred through the domain is increasing by increasing the velocity, which results in decrease of temperature near the heat source in the middle of the domain.

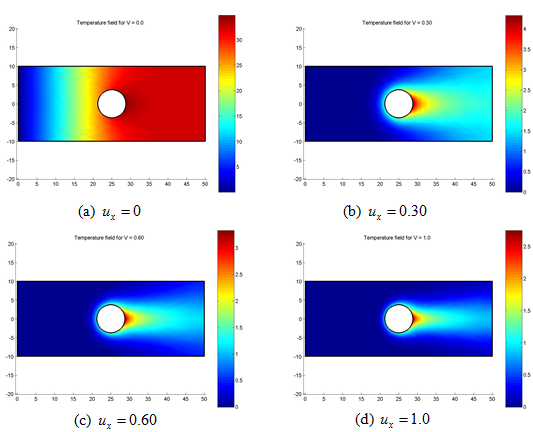


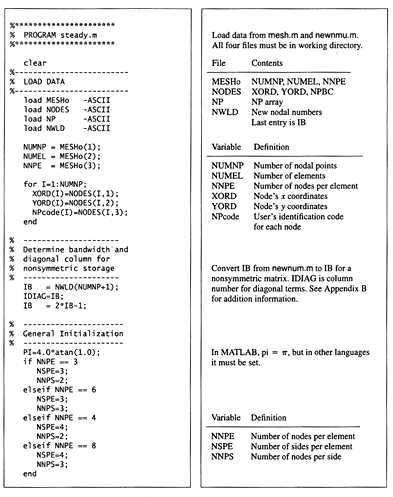
Figure 8. Temperature Flow Field for Different Values of Entrance Velocity

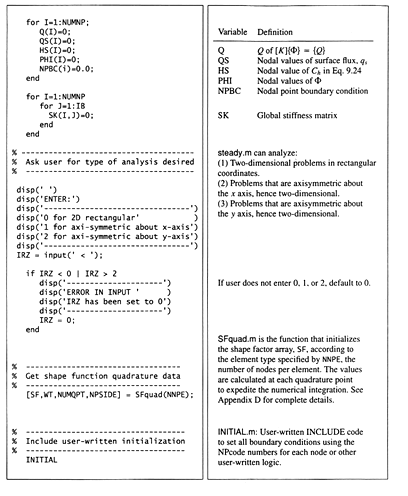
**3.3. *Conclusions***

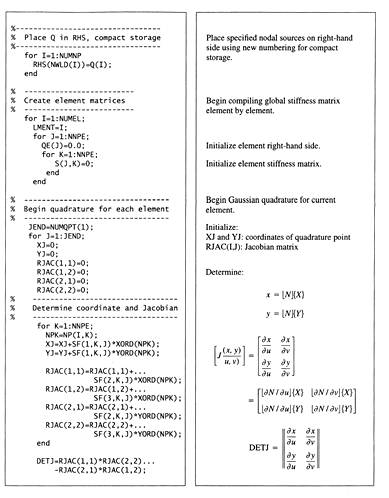
In this study, a two-dimensional non-viscous incompressible steady flow problem is analysed using the finite element method through solving partial differential equations of the fluid flow. The fluid flow is expressed by partial differential equation (Poisson’s equation). While, heat transfer is analyzed using the energy equation. The flow field of that fluid is then used to solve the partial differential equations of temperature flow. The domain of the problem is discretized to a large number of four-node is parametric elements to assure the accuracy of the solution. The influence of the presence of heat source inside the domain on the temperature flow field is also investigated. Analyses are done by studying both flow field and temperature field for various values of entrance velocity. Results showed that both fluid flow and temperature flow are influenced considerably by changing entrance velocity. Also, there is an obvious effect on the temperature flow field due the presence of an energy source in the middle of the domain.

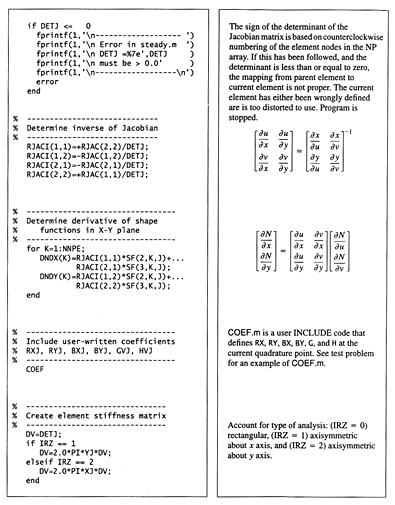
**Appendix I**

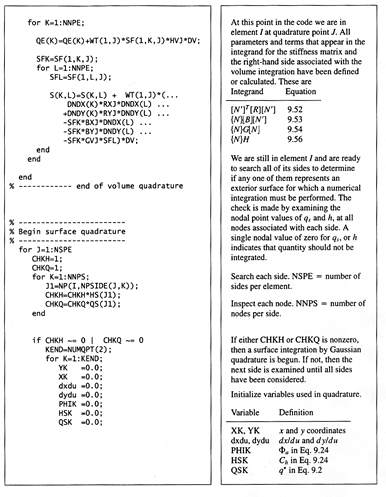
**(steady.m Matlab Code)**

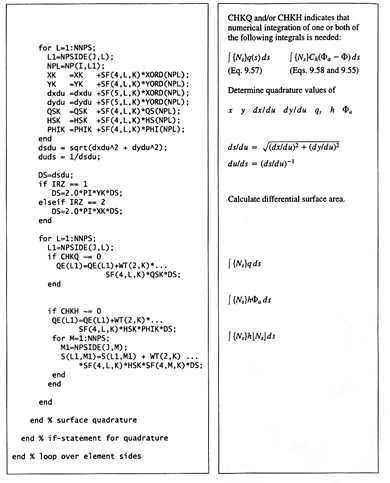


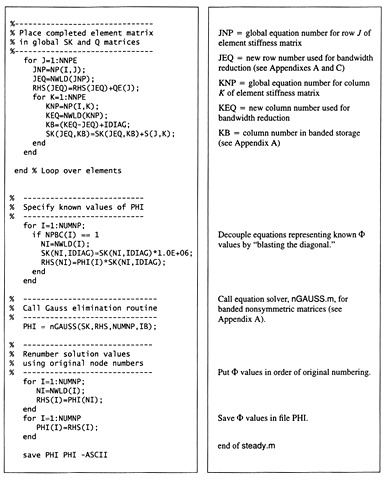












**Questions for self-control**

1. What is the governing equation of two dimensional flow?

2. What is the governing equation for fluid flow analysis used for?

3. Where does the governing equation for temperature flow analysis apply?

4. How is the temperature flow analysis performed?

5. How is the fluid flow analysis defined?

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**Lecture 9**

**Lecture topic: The model of a spherical shell.** **The buckling of a spherical shell**

**The plan**

**1. The model of a spherical shell**

1.1. The model. Bernoulli’s hypothesis. Displacement vector and strain tensor

1.2. Quasi-mean values. Equation of motion

1.3. Integrals of the stress elements. Quasi-moments

1.4. Integrals of displacement vector

1.5. Equation of motion in quasi-mean values

1.6. Quasi-mean value of the shell density. The differential equation

**2. The buckling of a spherical shell**

2.1. The buckling of a spherical shell

2.2. Load on the upper face. Stress on the lower surface of the shell

2.3. The differential equation of time dependent flexure

2.4. Spherical effects with respect to the plane plate

**1. The model of a spherical shell**

**1.1. *The model. Bernoulli’s hypothesis. Displacement vector and strain tensor***

A spherical elastic, homogeneous shell having the elastic moduli equal to  is considered and the usual spherical co-ordinate system  is used. However, some of the derived results are also valid in the case of a more general, non-elastic spherical shell. In the initial state, the homogeneous density is denoted by and the median spherical surface of the shell has the equation

, (1)

where  is the radius of the sphere. At a certain time during the deformation, the median surface will be

, (2)

where  is the flexure of the shell, positive downward. Hence the unit vector normal to the median surface at a certain point of coordinates is

. (3)

Neglecting the quantities of the second order, it follows that

. (4)

Consider the initial, non-deformed state of the shell and two points of co-ordinates  and , where H=2h is the thickness of the shell. It follows that the segment has the unit vector equal to . At an arbitrary time, in the deformed state of the shell, the point is displaced to a point having the position vector equal to

, (5)

while the point is displaced to the point having the position vector equal to

. (6)

Here,  is the displacement vector at a point of certain spherical co-ordinates. Assuming the shell is thin, quantities of the order are neglected and the segment  has the unit vector equal to

. (7)

The partial derivatives in (7) are computed at the point . It is assumed that Bernoulli’s hypothesis is valid for all time. It follows that a segment inside the shell, which is initially normal to the median spherical surface, will be always normal to the median surface during the deformation. From (4) and (7), it is supposed that the displacement vector has the elements

, (8)

where  and  are two unknown functions representing the horizontal displacements of the points initially placed on the median sphere. A further hypothesis on  and  will be later considered. It follows the elements of the strain tensor are

, (9)

, (10)

, (11)

, (12)

, (13)

and

. (14)

Hence the trace of the strain tensor is

, (15)

where the two-dimensional LAPLACE operator is

. (16)

**1.2. *Quasi-mean values. Equations of motion***

Consider the stress tensor in spherical coordinates. For each element of the tensor, the corresponding quasi-mean value is defined, for example, by

. (17)

In the same way, the corresponding quasi-moment is defined by

. (18)

The equations of motion in spherical coordinates are

. (19)

In the above equations,  denotes the density of the shell in the deformed state. Assumed to be a negligible second order effect, that density will be replaced by the initial density at the right side of (19).

The first equation in (19) will be multiplied by  and the quasi-mean operator defined by (17) will be applied. The next two equations in (19) will be multiplied by and the quasi-moment operator defined by (18) will be applied. In order to do that, some intermediary results are necessary.

**1.3. *Integrals of the stress elements. Quasi-moments***

Let the stress values on the upper/lower faces of the shell be denoted as

 (20)

and, respectively, by

. (21)

Elementary computations show that

. (22)

In the case of the elastic shell, the Hooke’s law leads to

 . (23)

Integrating by parts, it follows that

 (24)

Also,

. (25)

Similar relations are derived for .

Also,

. (26)

In the same way, it follows that

, (27)

and

. (28)

**1.4. *Integrals of the displacement vector***

Using (8), it follows

. (29)

Also,

, (30)

and

. (31)

**1.5. *Equations of motion in quasi-mean values***

Let the first equation in (19) be multiplied by . By applying the quasi-mean operator defined by (17), it follows

. (32)

The next two equations in (19) are multiplied by and the quasi-moment operator defined by (18) is applied. Hence

 (33)

and

. (34)

**1.6. *Quasi-mean value of the shell density. The differential equation***

According to (17), the quasi-mean of a constant is equal to that constant. Applying the quasi-mean operator to the mass balance equation in the linear approximation gives

. (35)

It will be further assumed that the quasi-mean of the density is constant and equal to its initial value. From (35) it follows that

, (36)

i.e., using (15),

. (37)

For the elastic shell,

. (38)

By using (9), it follows

, (39)

or, using (36),

. (40)

The term  will be neglected. Using (40) and substituting  and  between (32)-(34) it follows

(41)

A correction due to the compressive horizontal stresses ,acting at the ends of the shell, approximately along the andaxes respectively, follows to be further considered.

**2. The buckling of a spherical shell**

**2.1. *The buckling of a spherical shell***

Consider the element ABCD of the mean surface of the deformed shell from Fig.G1. It is centred at the point M, where the local unit vectors are . The centres of the lateral sides are denoted by . On the meridian cross section  is acting a normal compressive stress , having the approximate direction from West to East, and a tangential stress , having the approximate direction from North to South. On the parallel cross section  is acting a normal compressive stress , having the approximate direction from North to South, and a tangential stress , having the approximate direction from West to East. Let  be the angle between the normal vector to the meridian section  and the unit vector . Also, let  be the angle between the normal vector to the parallel section and the unit vector . The concentrated force acting at the point is

(42)

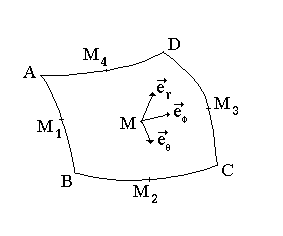


Fig. G1. A rectangular element of the mean surface of the deformed shell

In the same way, the concentrated force acting at the point  is equal to

(43)

The concentrated forces acting at the points  and  are respectively equal to

. (44)

But

. (45)

Let a surface density of forces normal to the element of the shell, having the same mechanical effect as the presence of the compressive stress. The force due to that density is equal to

. (46)

It follows the deformed element of the shell is into an equilibrium state due to the action of the lateral stress and to the opposite force , i.e.

. (47)

Hence the next three equations of equilibrium are obtained:

 (48)

, (49)

and

. (50)

For the particular case when  is a constant and , eqs.(48) - (50) show that the presence of the lateral compressive stress is equivalent to a supplemental load placed on the upper face of the shell, having the value

, (51)

a result similar to the case of the plane plate (Timoshenko and Woinowsky-Krieger 1959; Nowacki 1961). Quantities of the second order, like have been neglected again.

**2.2. *Load on the upper face. Stress on the lower surface of the shell***

Consider now the differential equation (41). Usually, the horizontal loads for the upper face of the shell are neglected, and it is assumed that

, (52)

where is the density of the filling sediments,  is the load and an inertial term is considered. For the material below the shell, the next constitutive equation is assumed

, (53)

where is a reference pressure, is the density of that material and is the unit tensor. The elastic coefficients are and the viscosity is denoted by. The strain tensor inside the material is and the strain rate here is .

The first boundary condition assumed on the lower face of the shell, having the equation , is the continuity of the displacement vector. Hence the elements of the displacement vector inside the material placed immediately below the lower face of the shell are assumed to be equal to the same elements at the points of the shell placed on the lower face. By using eqs. (8), it follows

. (54)

Hence the next values for the strain tensor immediately below the shell are obtained

. (55)

The normal vector at the lower face of the shell is . From Newton’s third low, it follows a relation between the stress inside the shell and the stress inside the material below the shell, i.e.

, (56)

on the lower surface of the shell. Here, the next elements of the stress are obtained after some elementary computations

, (57)

, (58)

and

. (59)

**2.3. *The differential equation of time dependent flexure***

The reference pressurein (53) is selected in order the flexure of the shell to vanish in the absence of the load. Substituting the loads on the upper/lower faces of the shell in eq. (41) and taking into account the presence of the lateral gstress, a generalization of the Sophie Germain plain plate static equation in the case of a time dependent flexure of a spherical elastic shell is obtained as

. (60)

If quantities of the order are neglected with respect to unit, it follows finally that

, (61)

where

 (62)

is the flexural rigidity and is the velocity of the P-wave through the shell. Also,

. (63)

**2.4. *Spherical effects with respect to the plane plate***

For the usual materials, . In that case, the corresponding equation for the plane plate (Ivan 1997) is

. (64)

With respect to (64), a change of the density difference according to (63) and a substitution of the real lateral stresses , by their apparent values can be observed in eq.(s61). A supplemental load is present according to (g63c). For usual values (e.g. Ivan 1997a) like , ,,, all these effects are usually negligible and difficult to be observed in real life. To compare to  in the left side of (61), the case of a rectangular plate having the sides equal to is considered. Here, the flexure is proportional to a product of sins (cosines) functions, i.e. . It follows the LAPLACE operator of the flexure is proportional to. For the fundamental mode () it follows that

. (65)

That ratio is negligible too in the usual cases.

It can be concluded that in the usual cases, the sphericity of the crustal plates can be ignored and the equation derived for the flat plate can be used.

**Questions for self-control**

1. What is the main meaning of Bernoulli's hypothesis?

2. How are quasi-mean values calculated?

3. How is the equation of motion in quasi-mean values written?

4. What spherical effects with respect to the plane plate do you know?

5. How is the differential equation of time dependent flexure written?

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**Lecture 10**

**Lecture topic: Linear and non-linear models**

**The plan**

**1. Linear models**

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**1. Linear models**

**1.1. *Introduction***

Especially for geological processes at a large time scale and great values of the stress, the internal friction of the material cannot be ignored. Consequently, the Hooke’s law has to be replaced by assuming different constitutive equations (models). Their expressions are mainly depending on the time scale of the geophysical process to be modelled. In relation to seismic or seismological applications, for example, short periods and short stresses are required (usually, seconds up to minutes, with a maximum value around one hour for the fundamental mode of the free oscillations of the Earth). Here, the non-elasticity is related to the very short period irreversible changes in the crystal defect structures of the medium (e.g. opening/closing of pre-existing cracks) and/or to the energy lost by friction at the two sides of a crack or on the non-elastic boundary coupling grain particles to the adjacent material (Aki and Richards, 1980; Ranalli, 1987; Wahr, 1996).

With respect to the mathematical relation between stress and strain, there are two kinds of constitutive equations (models).

**1.2. *Linear models***

Simplified models involves a linear relation between stress (and its derivatives of various orders with respect to time) and strain (together with its time derivatives).

In the beginning, only the 1-D case will be discussed. More general examples follow to be presented in relation with the dynamic aspects of the flexure of a plate (shell) and to the accretion prism. For each constitutive equation, a mechanical analogue can be considered. The elastic part will be represented by a spring, while the inelastic (viscous) behaviour is associated to a dashpot. Both parts are supposed to be linear ones, i.e. a linear relation is valid for the spring and a similar linear relation holds for the dashpot .

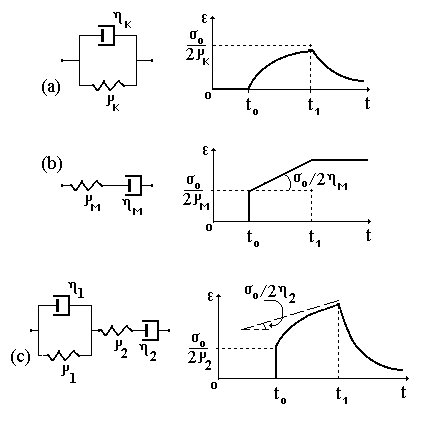


Fig. H1. (a) Kelvin-Voigt model; (b) Maxwell model; (c) Burgers model

**a) *Kelvin-Voigt (strong viscous) model***

The mechanical analogue of the first model to be considered is represented in Fig.H1a. The total stress is the sum between the stress in the spring and the stress in the dashpot, while the total deformation is equal to both the deformation of the spring and the deformation of the dashpot. It follows that Kelvin-Voigt model has the next constitutive equation

, (1)

where the dot shows the (total, material) derivative with respect to time. Here, the second term is the inelastic one, being the viscosity. Suppose now a constant stress equal to  is applied. Elementary computations show that the differential equation

 (2)

with the initial condition

 (3)

has the solution

, (4)

for . Hence, for very great values of time, the strain approaches a limiting value equal to

. (5)

The “flowage function”, denoted by

, (6)

shows that for a constant stress (equal to unit here), there is a temporal variation of the strain.

Suppose now at a certain moment , the constant stress  is removed, the corresponding strain at that moment being equal to . It follows now the corresponding solution decreases towards zero as

. (7)

**b) *Maxwell (viscous-elastic) model***

Consider the mechanical analogue represented in Fig. H1b. The total stress is equal to both the stress of the spring and to the stress of the dashpot, while the total deformation is the sum between the deformation of the spring and the deformation of the dashpot. It follows that Maxwell model has the next constitutive equation

, (8)

with the initial condition represented by (3).

Suppose again a constant stress equal to  is applied. The spring is instantly deformed to a value equal corresponding to the first term in the right hand of (8), i.e.

, (9)

and the solution of (8) (for a constant stress ) with the initial condition (9) is the straight line

, (10)

having the slope related to the stress  and to the viscosity of the dashpot. Suppose now at a certain moment , the constant stress  is removed, the corresponding strain at that moment being equal to . It follows from (8) that the strain remains constant.

**c) *Burgers (general linear) model***

The third model to be considered has the mechanical analogue represented in Fig.H1c.

**Exercise**. Show that the corresponding differential equation is

. (11)

Consider now the same initial condition (h3) and suppose again a constant stress equal to  is applied. In a similar manner to Maxwell model, the system is instantly deformed to a value equal to

 (12)

and the solution of (11) (for a constant stress ) with the initial condition (12) is

, (13)

where  is an unknown coefficient (because (11) is a second order differential equation, two initial conditions are required to obtain the complete solution). However, differentiating (13) it follows

. (14)

Hence, for great values of time, the solution (13) approaches asymptotically to a straight line having the slope equal to . Suppose now at a certain moment , the constant stress  is removed, the corresponding strain at that moment being equal to . It follows from (13) that the strain decreases exponentially towards zero, i.e

. (15)

**Exercise.** Show that two (or more) springs / dashpots connected in series (or parallel) sequence are equivalent to a single spring / dashpot. Justify that the Burgers model is the general linear model.

**d) *Remarks on the linear models***

In the most general case, the linear relation between stress and strain can be written as

, (16)

where

 (17)

and

 (18)

are formal polynomials of the variable  representing the derivative with respect to time, applied to stress and strain respectively. Here, are fourth rank tensors. For example, with respect to the Maxwell body having the constitutive equation (8), it follows that

. (19)

A common way to solve (16) is by using the Laplace transform (e.g. Sokolnikoff and Redheffer, 1958). Consider a certain function of one real variable, providing that

1. ;
2. is piecewise continuous on every finite interval;
3. there are two constants  in order to have  , for an arbitrary .

Under the above conditions, the Laplace transform of is a new function of the variable , defined by

. (20)

**Exercise 1.** Show that:

(a) The Laplace transform of the first derivative is

, (21)

1. . (22)
2. The *convolution theorem*. Consider the functions vanishing for negative values of their argument and a new function (also vanishing for negative values of the argument defined) by the *convolution product*

. (23)

Show that

. (24)

As an example, consider again the Maxwell model by applying the Laplace transform to both sides of (8). It follows that

, (25)

where  are the initial stress and strain respectively.

**Exercise 2.** By using the above properties of the Laplace transform, show that

. (26)

If the initial conditions are elastically coupled, i.e.

. (27)

it follows an integral representation of the stress which is independent on the initial conditions

. (28)

Hence, the actual value of the stress is related both to the actual value of the deformation and to the previous values of the strain (i.e. the stress depends on the “history” of the deformation). If a constant strain  is applied to the Maxwell body, it follows from (28) that the stress decreases as

, (29)

representing a “relaxation phenomenon” (stress decreases in time if a constant deformation is present). Here, the function

 (30)

is the “relaxation” kernel and the parameter  is the relaxation time.

A very similar approach (e.g. Wahr 1996) is based on the use of Fourier transform. Formally, the results derived by using Fourier transform are derived from the same results obtained with Laplace transform by performing the substitution .

The above 1-D models can be generalised for the 3-D case. For example, consider again the Maxwell body. There is a strong experimental evidence that the Maxwell Rheology applies only to the dissipation of the shear energy, i.e. the stress and strain tensors in (8) are the deviatoric tensors

. (31)

It will be further assumed that there is no dissipation of the compressional energy, i.e. a proportionality like

, (32)

is valid. By differentiating with respect to time

. (33)

Substituting (31) and (33) in (8) it follows

. (34)

Hence

. (35)

By applying the Fourier transform to both sides of (35), it follows

, (36)

i.e. a relation similar to Hooke’s law is valid between the Fourier (or Laplace) transforms of stress and strain. Here, the coefficients similar to Lamé parameters are

. (37)

At short periods  correspond high values of the pulsation

.

From (37) it follows

 (38)

and the behaviour of the Maxwell body is an elastic one.

At long periods  correspond low values of the pulsation and

 (39)

and the Maxwell body is a fluid having the compressional coefficient denoted by .

**2. Non-linear models**

**2.1. *Non-linear models***

For tectonic applications, large stresses and periods of thousands to millions of years are appropriate. Here, the non-elastic behaviour is probably related to the diffusion or dislocation creep of the molecules, a major factor being the high temperatures.

There are great difficulties to consider constitutive equations with non-linear relations between stress and strain, but some attempts have been made. A very common non-linear model is the work-hardening plasticity (e.g. Ranalli, 1994)

, (40)

where  is the component of the deviatoric stress,



is related to the second invariant of the deviatoric stress, and  are material parameters.

**2.2. *Brittle. Creep. Empirical criteria***

The usual materials are reacting in an elastic manner only for small values of the (deviatoric) stress, i.e. for stress values smaller than a limiting value representing the yield strength (or the yield stress), denoted by . The yield stress is a function of the nature of the material, of the temperature, pressure, the chemical composition of the adjacent rocks and, finally, of the history of the deformation (i.e. the intermediate steps followed to attend the yield value).

When the yield stress is attended, there are two possibilities of behaviour of the material:

* a rupture deformation of the rock, when the continuity of the deformation is lost, usually along a fault surface; this is the case of the *brittle materials*. The process is illustrated in Fig.H2a and b.
* a plastic, irreversibly flow of the material (*creep*), when, apparently, the continuity holds. The phenomenon is quite similar to the usual viscous flow, but it can be observed only when the yield stress is attended. This is the case of the *ductile materials.* The process is illustrated in Fig.H2c. An usual non-linear constitutive equation is the Byerlee power-law creep

, (41)

where are material parameters, is the activation enthalpy,  is the gas constant and  is the absolute temperature. It should be noted that the same material can act as a brittle or a ductile one according to the external conditions.

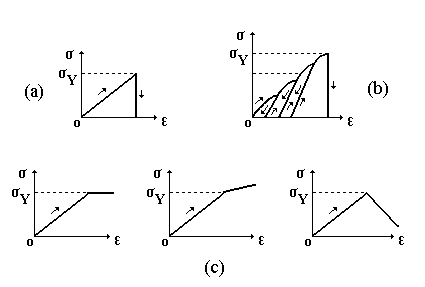


Fig. H2. (a) Faulting of a brittle material; (b) Increasing of the yield stress due to the history of deformation; (c) Creep of a ductile material.

**2.3. *Empirical criteria for shear-faulting. Tresca criterion. Coulomb-Navier criterion***

A first criterion also (valid for plasticity) is due to Tresca. It assumes that faulting (for brittle materials) or creep (for ductile ones) is attended at that points of the material where the maximum value of the shear (tangential) stress is equal to a yield value denoted by . Consider now a homogeneous (constant) stress field inside the material. Such a case can be obtained either by considering an infinitesimal volume of material or by taking into account a prismatic body with very large (infinite) sides. Consider the eigen-values of the stress tensor. It will be assumed that they are denoted in order to have , with the remark that ,in real life, stress is assumed to be positive for compression. Hence, with respect to Fig.H3, let and  in eqs. (11) and (13). It follows that

, (42)

where are so-called “normal stress” and “tangential (shear) stress” respectively, acting on a plane inside the material. The plane is at an angle  with the direction of the maximum compressive stress. The outer-pointing normal at that plane makes an angle  with the direction of the maximum compressive stress. With respect to a reference system, eqs. (42) are the parametric equations of the Mohr circle, plotted in Fig. H3. In the most general case, stress field is varying from point to point inside the material. Hence both the eigen-vectors of the stress tensor (i.e. the local directions of the maximum / minimum compressive stress) and the eigen-values of that tensor (i.e. the magnitudes of the maximum / minimum compressive stress) are also changing from point to point. For a fixed point inside the material, both normal stress and shear (tangential) stress are varying with the angle between the plane (with respect to normal and tangential stresses are defined) and the direction of the local maximum compressive stress. Consider a certain point inside the material and imagine various planes passing through that point. Hence, according to Tresca empirical criterion, failure of the material is produced here if

. (43)

Using eq. (42b), it follows

. (44)

Consider now a homogeneous stressed material subject to progressive increasing values of the difference . The material is characterised by a yield value denoted by . Eq. (44) shows that:

* if , there is no failure inside the material;
* when the equality

 (45)

is attended, a failure is produced along the planes at angles  with the direction of the maximum compressive stress.

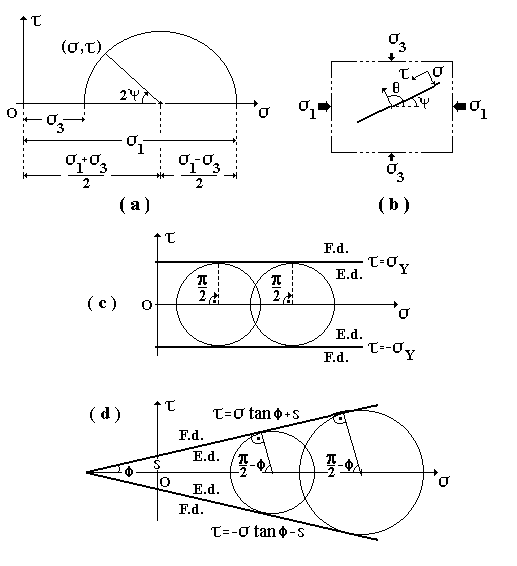


Fig. H3. (a) The Mohr circle (b) A plane inside the material, at an angle ψ with the direction of the maximum compressive stress; (c) Tresca criterion. E.d and F.D. denote the elastic domain and the failure domain respectively; (d) Coulomb-Navier criterion.

Eq. (45) represents the Tresca criterion in terms of the eigenvalues of the stress tensor. In the case of a material subject to a non-homogeneous stress field, eq. (45) is a local condition. Here, the eigen-values are obtained. For a compressive stress, that values are expected to be negative ones. They have to be denoted by , where .

A second criterion is due to Coulomb and Navier. It can be used to describe only the shear fracture. According to it, a material is characterised by the cohesive strength denoted by and by the coefficient of friction, denoted by . Here,  is the angle of internal friction ( in most rocks). According to Coulomb-Navier empirical criterion, shear failure of the material is produced at its points where

. (46)

Using eqs. (42), it follows

. (47)

If is a solution of (47),  (or just ) is a solution too. Hence, without loss of generality, the values of the angle will be limited to the first quadrant, where eq. (47) is

. (48)

Eq. (48) shows that:

* if , there is no failure of the material;
* when the equality

. (49)

is attended, a share fracture is produced along the planes at angles  with the direction of the maximum compressive stress. Eq. (49) represents the Coulomb-Navier criterion in terms of the eigenvalues of the stress tensor. Taking into account that

,

eq. (49) can be written as

, (50)

outlining that Coulomb-Navier criterion is a generalisation of Tresca criterion for a non-zero internal friction.

**2.4. *Von Mises-Hencky criterion for ductile flow (plasticity)***

Because the ductile (plastic) flow is independent of the co-ordinate system used, it depends only on the invariants of the stress tensor. Hence an equation like

, (51)

will be valid. There are strong experimental evidence that the plastic flow does not depend on the hydrostatic pressure, being also similar for compressive and tensile states of stress. It follows the function  in (51) depends only on the second invariant of the stress deviator (see eq. (23)). Hence, ductile flow occurs only at those points of material where the second invariant of the deviator stress reaches a certain value, depending on the nature of the material. Using eq. (24), the criterion of Von Mises-Hencky assumes that ductile flow occurs at those points of the material where

. (52)

In terms of the principal stresses, eq. (52) is

. (53)

Hence the criterion of Von Mises-Hencky can be regarded too as a generalisation of Tresca criterion, by taking into account the presence of the intermediate stress.

**2.5. *Rheological models***

**a) *Saint-Venant body (elastic-plastic material)***

The behaviour of that material is characterised by linear elasticity for stress values below the yield strength. When the yield stress is attended, the body exhibits a pure plasticity. Its constitutive equation (using deviator tensors) has the symbolic form

. (54)

The above material has the mechanical analogue presented in Fig. (H4a), being referred as a Saint-Venant body.

1. ***Bingham body (visco-plastic material)***

Similar to the Saint-Venant body, that material exhibits linear elasticity for stress values lower than the yield strength, but flows linearly above that value. The strain rate is proportional to the difference between the deviatoric stress and the yield strength. Its constitutive equation is

. (55)

The above material has the mechanical analogue presented in Fig. (H4b), being referred as a Bingham body.

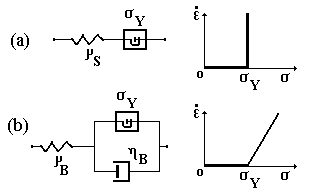


Fig. H4. (a) The Saint-Venant body; (b) The Bingham body

**2.6. *The accretion wedge. The model***

Consider a 2-D prismatic body having a triangular vertical section (Fig.I1), in the presence of gravity. The wedge rests on a rigid basement having the slope equal to . Both the compressional force acting on the left side of the wedge and the friction to the basement cause thickening of the incompressible material and the development of a topographical slope equal to . It is assumed that the material is into a state of plastic yielding according to the Von Mises-Hencky criterion. It follows to obtain a condition relating the slopes of the topography and that of the basement to the geometry of the wedge, its yield strength and the friction coefficient to the basement.

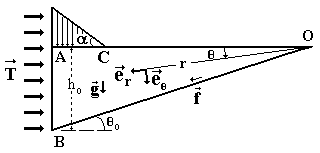


Fig. I1. The 2-D accretion wedge

**2.7. *Equations of equilibrium. Yield condition. Stress field***

Taking into account that the 2-D case is discussed, the stress in polar co-ordinates is

. (56)

Because the material is assumed to be incompressible, the Poisson coefficient is .

Using polar coordinates, the equilibrium equation (54) - (55) in the presence of gravity are

, (57)

. (58)

Using (56), the yield condition (53) is

.

Taking into account again that the compressive stress is assumed to have positive sign, eqs. (57) - (58) give

, (59)

where the trace of the stress has been denoted by and  is the angle between the radius and the local direction of the maximum compressive stress (variable inside the wedge). However, it will be assumed that only. After some manipulations, substituting (59) into (57) - (58) gives

 (60)

Eqs. (60a) and (60b) are differentiated with respect to  respectively, the results of the differentiation being equal each other. After some elementary manipulations, it follows that

,

or

, (61)

where C is a constant of integration. Substituting (61) into (60a) gives

,

or

, (62)

To find the unknown function , eq. (62) is substituted into (60b) to obtain

.

Using (61), it follows that

,

i.e.

,

where A is another constant of integration. Hence the final stress inside the wedge is

. (63)

**2.8. *Boundary conditions. Final results***

Consider the segment AC placed on the side OA of the wedge, having and , where the point C is very closed to the point A. The outward pointing normal vector is. Here, is acting the lithostatic pressure due to the topography. Hence

, (64)

i.e.

. (65)

Because the angle has very small values, it will be assumed for all angles that

,

or

. (66)

However, all the next derivations are supposed to be valid at the rear of the wedge, where the topography is generated due to the horizontal compression, i.e. the radius is a mean value of the lengths OA and OC, the point C being closed to A. By differentiating (63b), eq. (66) leads to

, (67)

where

. (68)

Consider now the side OB of the wedge, having and the outward pointing normal vector . Here, is acting the friction force due to the basement, assumed to have the magnitude equal to , where is a friction coefficient. Hence

,

or

. (69)

The next partial derivative follows to be evaluated in two ways. In the first approach, eqs. (69), (65) and (68) are used to give

. (70)

The dominant stress is into the wedge is the horizontal compression. Hence , both angles having small values. It follows . Eqs.(64), (61), (679) and (68) give

. (71)

From (70) and (71) it follows

, (72)

showing that the friction force (resistance to sliding of the wedge onto the basement) is balanced by tqo forces. The first one is due to the topography and the second force is related to the compressive stress and to the slope of the basement. Further details related to the application of eq. (72) in real cases are presented by Ranalli (1987).

**Questions for self-control**

1. Which models are linear?

2. Which models are non-linear?

3. What are the basics of the Kelvin-Voigt (strong viscous) model?

4. How is the Maxwell (viscous-elastic) model defined?

5. How is the Burgers (general linear) model calculated?

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1. [↑](#footnote-ref-1)
2. For the sake of simplicity, we limit ourselves here to two-dimensional cases. [↑](#footnote-ref-2)
3. This concerns Galerkin's approximation. For advection dominated problems, other weighting function have to be used. [↑](#footnote-ref-3)