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**The course of lectures**

**by the discipline «** **Lagrangian mechanics and nonholonomic systems**

**(in English)»**

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Karaganda 2022

**Lecture 1**

**Lecture topic:** [**Lagrangian Mechanics**](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_chap_1)

**The plan**

1. [Introduction](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_chap_Temp_2)

2. [Lagrangian Mechanics](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_chap_1)

## [3. The Principle of Stationary Action](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.1)

## [4. Configuration Spaces](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.2)

**1.** [**Introduction**](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_chap_Temp_2)

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There has been a remarkable revival of interest in classical mechanics in recent years. We now know that there is much more to classical mechanics than previously suspected. The behavior of classical systems is surprisingly rich; derivation of the equations of motion, the focus of traditional presentations of mechanics, is just the beginning. Classical systems display a complicated array of phenomena such as nonlinear resonances, chaotic behavior, and transitions to chaos.

Traditional treatments of mechanics concentrate most of their effort on the extremely small class of symbolically tractable dynamical systems. We concentrate on developing general methods for studying the behavior of systems, whether or not they have a symbolic solution. Typical systems exhibit behavior that is qualitatively different from the solvable systems and surprisingly complicated. We focus on the phenomena of motion, and we make extensive use of computer simulation to explore this motion.

Even when a system is not symbolically tractable, the tools of modern dynamics allow one to extract a qualitative understanding. Rather than concentrating on symbolic descriptions, we concentrate on geometric features of the set of possible trajectories. Such tools provide a basis for the systematic analysis of numerical or experimental data.

Classical mechanics is deceptively simple. It is surprisingly easy to get the right answer with fallacious reasoning or without real understanding. Traditional mathematical notation contributes to this problem. Symbols have ambiguous meanings that depend on context, and often even change within a given context.[1](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-5.html%22%20%5Cl%20%22footnote_Temp_3) For example, a fundamental result of mechanics is the Lagrange equations. In traditional notation the Lagrange equations are written



The Lagrangian *L* must be interpreted as a function of the position and velocity components *qi* and *i*, so that the partial derivatives make sense, but then in order for the time derivative *d*/*dt* to make sense solution paths must have been inserted into the partial derivatives of the Lagrangian to make functions of time. The traditional use of ambiguous notation is convenient in simple situations, but in more complicated situations it can be a serious handicap to clear reasoning. In order that the reasoning be clear and unambiguous, we have adopted a more precise mathematical notation. Our notation is functional and follows that of modern mathematical presentations.[2](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-5.html%22%20%5Cl%20%22footnote_Temp_4) An introduction to our functional notation is in an appendix.

Computation also enters into the presentation of the mathematical ideas underlying mechanics. We require that our mathematical notations be explicit and precise enough that they can be interpreted automatically, as by a computer. As a consequence of this requirement the formulas and equations that appear in the text stand on their own. They have clear meaning, independent of the informal context. For example, we write Lagrange's equations in functional notation as follows:[3](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-5.html%22%20%5Cl%20%22footnote_Temp_5)



The Lagrangian *L* is a real-valued function of time *t*, coordinates *x*, and velocities *v*; the value is *L*(*t*, *x*, *v*). Partial derivatives are indicated as derivatives of functions with respect to particular argument positions; 2 *L* indicates the function obtained by taking the partial derivative of the Lagrangian function *L* with respect to the velocity argument position. The traditional partial derivative notation, which employs a derivative with respect to a ``variable,'' depends on context and can lead to ambiguity.[4](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-5.html%22%20%5Cl%20%22footnote_Temp_6) The partial derivatives of the Lagrangian are then explicitly evaluated along a path function *q*. The time derivative is taken and the Lagrange equations formed. Each step is explicit; there are no implicit substitutions.

Even in the introductory computer science class we never formally teach the language, because we do not have to. We just use it, and students pick it up in a few days. This is one great advantage of Lisp-like languages: They have very few ways of forming compound expressions, and almost no syntactic structure. All of the formal properties can be covered in an hour, like the rules of chess. After a short time we forget about the syntactic details of the language (because there are none) and get on with the real issues -- figuring out what we want to compute.

The advantage of Scheme over other languages for the exposition of classical mechanics is that the manipulation of procedures that implement mathematical functions is easier and more natural in Scheme than in other computer languages. Indeed, many theorems of mechanics are directly representable as Scheme programs.

The version of Scheme that we use in this book is MIT Scheme, augmented with a large library of software called Scmutils that extends the Scheme operators to be generic over a variety of mathematical objects, including symbolic expressions. The Scmutils library also provides support for the numerical methods we use in this book, such as quadrature, integration of systems of differential equations, and multivariate minimization.

The Scheme system, augmented with the Scmutils library, is free software. We provide this system, complete with documentation and source code, in a form that can be used with the GNU/Linux operating system, on the Internet at
http://www-mitpress.mit.edu/sicm.

When we started we expected that using this approach to formulate mechanics would be easy. We quickly learned that many things we thought we understood we did not in fact understand. Our requirement that our mathematical notations be explicit and precise enough that they can be interpreted automatically, as by a computer, is very effective in uncovering puns and flaws in reasoning. The resulting struggle to make the mathematics precise, yet clear and computationally effective, lasted far longer than we anticipated. We learned a great deal about both mechanics and computation by this process. We hope others, especially our competitors, will adopt these methods, which enhance understanding while slowing research.

**2.** [**Lagrangian Mechanics**](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_chap_1)

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The subject of this lecture is motion and the mathematical tools used to describe it.

Centuries of careful observations of the motions of the planets revealed regularities in those motions, allowing accurate predictions of phenomena such as eclipses and conjunctions. The effort to formulate these regularities and ultimately to understand them led to the development of mathematics and to the discovery that mathematics could be effectively used to describe aspects of the physical world. That mathematics can be used to describe natural phenomena is a remarkable fact.

A pin thrown by a juggler takes a rather predictable path and rotates in a rather predictable way. In fact, the skill of juggling depends crucially on this predictability. It is also a remarkable discovery that the same mathematical tools used to describe the motions of the planets can be used to describe the motion of the juggling pin.

Classical mechanics describes the motion of a system of particles, subject to forces describing their interactions. Complex physical objects, such as juggling pins, can be modeled as myriad particles with fixed spatial relationships maintained by stiff forces of interaction.

There are many conceivable ways a system could move that never occur. We can imagine that the juggling pin might pause in midair or go fourteen times around the head of the juggler before being caught, but these motions do not happen. How can we distinguish motions of a system that can actually occur from other conceivable motions? Perhaps we can invent some mathematical function that allows us to distinguish realizable motions from among all conceivable motions.

The motion of a system can be described by giving the position of every piece of the system at each moment. Such a description of the motion of the system is called a *configuration path*; the configuration path specifies the configuration as a function of time. The juggling pin rotates as it flies through the air; the configuration of the juggling pin is specified by giving the position and orientation of the pin. The motion of the juggling pin is specified by giving the position and orientation of the pin as a function of time.

The function that we seek takes a configuration path as an input and produces some output. We want this function to have some characteristic behavior when the input is a realizable path. For example, the output could be a number, and we could try to arrange that this number be zero only on realizable paths. Newton's equations of motion are of this form; at each moment Newton's differential equations must be satisfied.

However, there is an alternate strategy that provides more insight and power: we could look for a path-distinguishing function that has a minimum on the realizable paths -- on nearby unrealizable paths the value of the function is higher than it is on the realizable path. This is the *variational strategy*: for each physical system we invent a path-distinguishing function that distinguishes realizable motions of the system by having a stationary point for each realizable path.[1](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-7.html%22%20%5Cl%20%22footnote_Temp_8) For a great variety of systems realizable motions of the system can be formulated in terms of a variational principle.[2](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-7.html%22%20%5Cl%20%22footnote_Temp_9)

Mechanics, as invented by Newton and others of his era, describes the motion of a system in terms of the positions, velocities, and accelerations of each of the particles in the system. In contrast to the Newtonian formulation of mechanics, the variational formulation of mechanics describes the motion of a system in terms of aggregate quantities that are associated with the motion of the system as a whole.

In the Newtonian formulation the forces can often be written as derivatives of the potential energy of the system. The motion of the system is determined by considering how the individual component particles respond to these forces. The Newtonian formulation of the equations of motion is intrinsically a particle-by-particle description.

In the variational formulation the equations of motion are formulated in terms of the difference of the kinetic energy and the potential energy. The potential energy is a number that is characteristic of the arrangement of the particles in the system; the kinetic energy is a number that is determined by the velocities of the particles in the system. Neither the potential energy nor the kinetic energy depends on how those positions and velocities are specified. The difference is characteristic of the system as a whole and does not depend on the details of how the system is specified. So we are free to choose ways of describing the system that are easy to work with; we are liberated from the particle-by-particle description inherent in the Newtonian formulation.

The variational formulation has numerous advantages over the Newtonian formulation. The equations of motion for those parameters that describe the state of the system are derived in the same way regardless of the choice of those parameters: the method of formulation does not depend on the choice of coordinate system. If there are positional constraints among the particles of a system the Newtonian formulation requires that we consider the forces maintaining these constraints, whereas in the variational formulation the constraints can be built into the coordinates. The variational formulation reveals the association of conservation laws with symmetries. The variational formulation provides a framework for placing any particular motion of a system in the context of all possible motions of the system. We pursue the variational formulation because of these advantages.

## [3. The Principle of Stationary Action](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.1)

Let us suppose that for each physical system there is a path-distinguishing function that is stationary on realizable paths. We will try to deduce some of its properties.

#### [Experience of motion](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html%22%20%5Cl%20%22%25_toc_%25_sec_Temp_10)

Our ordinary experience suggests that physical motion can be described by configuration paths that are continuous and smooth.[3](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-8.html%22%20%5Cl%20%22footnote_Temp_11) We do not see the juggling pin jump from one place to another. Nor do we see the juggling pin suddenly change the way it is moving.

Our ordinary experience suggests that the motion of physical systems does not depend upon the entire history of the system. If we enter the room after the juggling pin has been thrown into the air we cannot tell when it left the juggler's hand. The juggler could have thrown the pin from a variety of places at a variety of times with the same apparent result as we walk through the door.[4](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-8.html%22%20%5Cl%20%22footnote_Temp_12) So the motion of the pin does not depend on the details of the history.

Our ordinary experience suggests that the motion of physical systems is deterministic. In fact, a small number of parameters summarize the important aspects of the history of the system and determine its future evolution. For example, at any moment the position, velocity, orientation, and rate of change of the orientation of the juggling pin are enough to completely determine the future motion of the pin.

#### [Realizable paths](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html%22%20%5Cl%20%22%25_toc_%25_sec_Temp_13)

From our experience of motion we develop certain expectations about realizable configuration paths. If a path is realizable, then any segment of the path is a realizable path segment. Conversely, a path is realizable if every segment of the path is a realizable path segment. The realizability of a path segment depends on all points of the path in the segment. The realizability of a path segment depends on every point of the path segment in the same way; no part of the path is special. The realizability of a path segment depends only on points of the path within the segment; the realizability of a path segment is a local property.

So we will try to arrange that the path-distinguishing function, constructed as an integral of a local property along the path, assumes an extreme value for any realizable path. Such a path-distinguishing function is traditionally called an action for the system. We use the word ``action'' to be consistent with common usage. Perhaps it would be clearer to continue to call it ``path-distinguishing function,'' but then it would be more difficult for others to know what we were talking about.[6](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-8.html%22%20%5Cl%20%22footnote_Temp_15)

In order to pursue the agenda of variational mechanics, we must invent action functions that are stationary on the realizable trajectories of the systems we are studying. We will consider actions that are integrals of some local property of the configuration path at each moment. Let be the configuration-path function; (t) is the configuration at time t. The action of the segment of the path in the time interval from t1 to t2 is[7](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-8.html%22%20%5Cl%20%22footnote_Temp_16)



where [] is a function of time that measures some local property of the path. It may depend upon the value of the function at that time and the value of any derivatives of at that time.[8](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-8.html%22%20%5Cl%20%22footnote_Temp_17)

The function measures some local property of the configuration path . We can decompose [] into two parts: a part that measures some property of a local description and a part that extracts a local description of the path from the path function. The function that measures the local property of the system depends on the particular physical system; the method of construction of a local description of a path from a path is the same for any system. We can write [] as a composition of these two functions:[10](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-8.html%22%20%5Cl%20%22footnote_Temp_19)



The function takes the path and produces a function of time whose value is an ordered tuple containing the time, the configuration at that time, the rate of change of the configuration at that time, and the values of higher derivatives of the path evaluated at that time. For the path and time t:[11](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-8.html%22%20%5Cl%20%22footnote_Temp_20)



We refer to this tuple, which includes as many derivatives as are needed, as the local tuple.

The function depends on the specific details of the physical system being investigated, but does not depend on any particular configuration path. The function computes a real-valued local property of the path. We will find that needs only a finite number of components of the local tuple to compute this property: The path can be locally reconstructed from the full local description; that depends on a finite number of components of the local tuple guarantees that it measures a local property.[12](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-8.html%22%20%5Cl%20%22footnote_Temp_21)

The advantage of this decomposition is that the local description of the path is computed by a uniform process from the configuration path, independent of the system being considered. All of the system-specific information is captured in the function .

The function is called a Lagrangian[13](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-8.html%22%20%5Cl%20%22footnote_Temp_22) for the system, and the resulting action,



is called the Lagrangian action. Lagrangians can be found for a great variety of systems. We will see that for many systems the Lagrangian can be taken to be the difference between kinetic and potential energy. Such Lagrangians depend only on the time, the configuration, and the rate of change of the configuration. We will focus on this class of systems, but will also consider more general systems from time to time.

The principle of stationary action[14](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-8.html%22%20%5Cl%20%22footnote_Temp_23) asserts that for each dynamical system we can cook up a Lagrangian such that a realizable path connecting the configurations at two times t1 and t2 is distinguished from all conceivable paths by the fact that the action [](t1, t2) is stationary with respect to variations of the path. For Lagrangians that depend only on the configuration and rate of change of configuration, the variations are restricted to those that preserve the configurations at t1 and t2.[15](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-8.html%22%20%5Cl%20%22footnote_Temp_24)

**Exercise 1.1.**  **Fermat optics**

Fermat observed that the laws of reflection and refraction could be accounted for by the following facts: Light travels in a straight line in any particular medium with a velocity that depends upon the medium. The path taken by a ray from a source to a destination through any sequence of media is a path of least total time, compared to neighboring paths. Show that these facts imply the laws of reflection and refraction.[16](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-8.html%22%20%5Cl%20%22footnote_Temp_26)

## [4. Configuration Spaces](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.2)

Let us consider mechanical systems that can be thought of as composed of constituent point particles, with mass and position, but with no internal structure.[17](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-9.html%22%20%5Cl%20%22footnote_Temp_27) Extended bodies maintain their shape because of spatial constraints among the constituent particles. Specifying the position of all the constituent particles of a system specifies the configuration of the system. The existence of constraints among parts of the system, such as those that determine the shape of an extended body, means that the constituent particles cannot assume all possible positions. The set of all configurations of the system that can be assumed is called the configuration space of the system. The dimension of the configuration space is the smallest number of parameters that have to be given to completely specify a configuration. The dimension of the configuration space is also called the number of degrees of freedom of the system.[18](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-9.html%22%20%5Cl%20%22footnote_Temp_28)

As a system evolves with time, the constituent particles move subject to the constraints. The motion of each constituent particle is specified by describing the changing configuration. Thus, the motion of the system may be described as evolving along a path in configuration space. The configuration path may be specified by a function, the configuration-path function, which gives the configuration of the system at any time.

**Exercise 1.2.**  **Degrees of freedom**

For each of the mechanical systems described below, give the number of degrees of freedom of the configuration space.

**a**.  Three juggling pins.

**b**.  A spherical pendulum, consisting of a point mass hanging from a rigid massless rod attached to a fixed support point. The pendulum bob may move in any direction subject to the constraint imposed by the rigid rod. The point mass is subject to the uniform force of gravity.

**c**.  A spherical double pendulum, consisting of one point mass hanging from a rigid massless rod attached to a second point mass hanging from a second massless rod attached to a fixed support point. The point mass is subject to the uniform force of gravity.

**d**.  A point mass sliding without friction on a rigid curved wire.

**e**.  A top consisting of a rigid axisymmetric body with one point on the symmetry axis of the body attached to a fixed support, subject to a uniform gravitational force.

**f**.  The same as **e**, but not axisymmetric.

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**Lecture 2**

## Lecture topic: [The Euler-Lagrange Equations](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.5)

**The plan**

## [1. The Euler-Lagrange Equations](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.5)

### [2. Derivation of the Lagrange Equations](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.5.1)

### [3. Computing Lagrange's Equations](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.5.2)

## [1. The Euler-Lagrange Equations](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.5)

The principle of stationary action characterizes the realizable paths of systems in configuration space as those for which the action has a stationary value. In elementary calculus, we learn that the critical points of a function are the points where the derivative vanishes. In an analogous way, the paths along which the action is stationary are solutions of a system of differential equations. This system, called the Euler-Lagrange equations or just the Lagrange equations, is the link that permits us to use the principle of stationary action to compute the motions of mechanical systems, and to relate the variational and Newtonian formulations of mechanics.

#### [Lagrange equations](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html%22%20%5Cl%20%22%25_toc_%25_sec_Temp_67)

We will find that if L is a Lagrangian for a system that depends on time, coordinates, and velocities, and if q is a coordinate path for which the action S[q](t1, t2) is stationary (with respect to any variation in the path that keeps the endpoints of the path fixed), then



Here L is a real-valued function of a local tuple; 1 L and 2 L denote the partial derivatives of L with respect to its generalized position and generalized velocity arguments.[49](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html%22%20%5Cl%20%22footnote_Temp_68) The function 2 L maps a local tuple to a structure whose components are the derivatives of L with respect to each component of the generalized velocity. The function [q] maps time to the local tuple:

[q](t) = ( t, q(t), Dq(t), ... ).

Thus the compositions 1 L o [q] and 2 L o [q] are functions of one argument, time. The Lagrange equations assert that the derivative of 2 L o [q] is equal to 1 L o [q], at any time. Given a Lagrangian, the Lagrange equations form a system of ordinary differential equations that must be satisfied by realizable paths.

### [2. Derivation of the Lagrange Equations](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html%22%20%5Cl%20%22%25_toc_%25_sec_1.5.1)

We will show that principle of stationary action implies that realizable paths satisfy a set of ordinary differential equations. First we will develop tools for investigating how path-dependent functions vary as the paths are varied. We will then apply these tools to the action, to derive the Lagrange equations.

#### [Varying a path](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html%22%20%5Cl%20%22%25_toc_%25_sec_Temp_70)

Suppose that we have a function f[q] that depends on a path q. How does the function vary as the path is varied? Let q be a coordinate path and q + be a varied path, where the function is a path-like function that can be added to the path q, and the factor is a scale factor. We define the variation f[q] of the function f on the path q by



The variation of f is a linear approximation to the change in the function f for small variations in the path. The variation of f depends on .

A simple example is the variation of the identity path function: I[q] = q. Applying the definition, we find



It is traditional to write I[q] simply as q. Another example is the variation of the path function that returns the derivative of the path. We have



It is traditional to write g[q] as Dq.

The variation may be represented in terms of a derivative. Let g() = f[q + ]; then



Variations have the following derivative-like properties. For path-dependent functions f and g and constant c:



Let F be a path-independent function and g be a path-dependent function; then



The operators D (differentiation) and (variation) commute in the following sense:



Variations also commute with integration in a similar sense.

If a path-dependent function f is stationary for a particular path q with respect to small changes in that path, then it must be stationary for a subset of those variations that results from adding small multiples of a particular function to q. So the statement

f[q] = 0

for arbitrary implies the function f is stationary for small variations of the path around q.

**Exercise 1.7.**  **Properties of **

Show that has the properties [1.23](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html#EQUATION_1.23)-[1.27](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html#EQUATION_1.27).

**Exercise 1.8.**  **Implementation of **

**a**.  Suppose we have a procedure f that implements a path-dependent function: for path q and time t it has the value ((f q) t). The procedure delta computes the variation ( f)[q](t) as the value of the expression ((((delta eta) f) q) t). Complete the definition of delta:

**b**.  Use your delta procedure to verify the properties of listed in exercise [1.7](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html#%_thm_1.7) for simple functions such as implemented by the procedure f:

This implements a one-degree-of-freedom path-dependent function that depends on the local tuple of the path at each moment. You should compute both sides of the equalities and compare the results.

#### [Varying the action](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html%22%20%5Cl%20%22%25_toc_%25_sec_Temp_74)

The action is the integral of the Lagrangian along a path:



For a realizable path q the variation of the action with respect to any variation that preserves the endpoints,

(t1) = (t2) = 0,

is zero:



Variation commutes with integration, so the variation of the action is



Using the fact that



which follows from equations ([1.20](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html#EQUATION_1.20)) and ([1.21](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html#EQUATION_1.21)), and using the chain rule for variations ([1.26](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html#EQUATION_1.26)), we get



Integrating the last term of equation ([1.32](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html#EQUATION_1.32)) by parts gives



For our variation we have (t1) = (t2) = 0, so the first term vanishes.

Thus the variation of the action is zero if and only if



The variation of the action is zero because, by assumption, q is a realizable path. Thus ([1.34](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html#EQUATION_1.34)) must be true for any function that is zero at the endpoints.

So we may conclude that the factor in curly brackets is identically zero:



This is just what we set out to obtain, the Lagrange equations.

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#### [Harmonic oscillator](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_Temp_77)

For an example, consider the harmonic oscillator. A Lagrangian is



Then



The Lagrangian is applied to a tuple of the time, a coordinate, and a velocity. The symbols t, x, and v are arbitrary; they are used to specify formal parameters of the Lagrangian.

Now suppose we have a configuration path y, which gives the coordinate of the oscillator y(t) for each time t. The initial segment of the corresponding local tuple at time t is



So



and



so the Lagrange equation is



which is the equation of motion of the harmonic oscillator.

#### [Orbital motion](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html%22%20%5Cl%20%22%25_toc_%25_sec_Temp_78)

As another example, consider the two-dimensional motion of a particle of mass m with gravitational potential energy - µ/r, where r is the distance to the center of attraction. A Lagrangian is[54](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html%22%20%5Cl%20%22footnote_Temp_79)



where and  are formal parameters for rectangular coordinates of the particle, and v and v are formal parameters for corresponding rectangular velocity components. Then[55](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html%22%20%5Cl%20%22footnote_Temp_80)



Similarly,



Now suppose we have a configuration path q = ( x, y ), so that the coordinate tuple at time t is

q(t) = ( x(t), y(t) ).

The initial segment of the local tuple at time t is



So



and



The component Lagrange equations at time t are



**Exercise 1.9.**  **Lagrange's equations**

Derive the Lagrange equations for the following systems, showing all of the intermediate steps as in the harmonic oscillator and orbital motion examples.

**a**.  A particle of mass m moves in a two-dimensional potential V(x, y) = (x2 + y2)/2 + x2 y - y3/3, where x and y are rectangular coordinates of the particle. A Lagrangian is

L(t; x, y; vx, vy) = (1/2) m (vx2 + vy2) - V(x, y).

**b**.  An ideal planar pendulum consists of a bob of mass m connected to a pivot by a massless rod of length l subject to uniform gravitational acceleration g. A Lagrangian is L(t, , ) = (1/2) m l2 2 + m g l cos . The formal parameters of L are t, , and ; measures the angle of the pendulum rod to a plumb line and is the angular velocity of the rod.[56](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html%22%20%5Cl%20%22footnote_Temp_82)

**c**.  A Lagrangian for a particle of mass m constrained to move on a sphere of radius R is

L(t; , ; , *ß*) = (1/2) m R2 (2 + (*ß* sin )2).

The angle is the colatitude of the particle and is the longitude; the rate of change of the colatitude is and the rate of change of the longitude is *ß*.

**Exercise 1.10.**  **Higher-derivative Lagrangians**

Derive Lagrange's equations for Lagrangians that depend on accelerations. In particular, show that the Lagrange equations for Lagrangians of the form L(t, q, , ) with terms are[57](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html%22%20%5Cl%20%22footnote_Temp_84)



In general, these equations, first derived by Poisson, will involve the fourth derivative of q. Note that the derivation is completely analogous to the derivation of the Lagrange equations without accelerations; it is just longer. What restrictions must we place on the variations so that the critical path satisfies a differential equation?

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### [3. Computing Lagrange's Equations](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.5.2)

The procedure for computing Lagrange's equations mirrors the functional expression ([1.18](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html#EQUATION_1.18)), where the procedure Gamma implements :

The argument of Lagrange-equations is a procedure that computes a Lagrangian. It returns a procedure that when applied to a path q returns a procedure of one argument (time) that computes the left-hand side of the Lagrange equations ([1.18](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html#EQUATION_1.18)). These residual values are zero if q is a path for which the Lagrangian action is stationary.

Observe that the Lagrange-equations procedure, like the Lagrange equations themselves, is valid for any generalized coordinate system. When we write programs to investigate particular systems, the procedures that implement the Lagrangian function and the path q will reflect the actual coordinates chosen to represent the system, but we use the same Lagrange-equations procedure in each case. This abstraction reflects the important fact that the method of derivation of Lagrange's equations from a Lagrangian is always the same; it is independent of the number of degrees of freedom, the topology of the configuration space, and the coordinate system used to describe points in the configuration space.

#### [The free particle](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html%22%20%5Cl%20%22%25_toc_%25_sec_Temp_86)

Consider again the case of a free particle. The Lagrangian is implemented by the procedure L-free-particle. Rather than numerically integrating and minimizing the action, as we did in section [1.4](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-11.html#%_sec_1.4), we can check Lagrange's equations for an arbitrary straight-line path

t ( at + a0, bt + b0, ct + c0 ):

That the residuals are zero indicates that the test path satisfies the Lagrange equations.

We can also apply the Lagrange-equations procedure to an arbitrary function:

The result is an expression containing the arbitrary time t and mass m, so it is zero precisely when D2 x = 0, which is the expected equation for a free particle.

#### [The harmonic oscillator](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html%22%20%5Cl%20%22%25_toc_%25_sec_Temp_89)

Consider the harmonic oscillator again, with Lagrangian ([1.16](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-11.html#EQUATION_1.16)). We know that the motion of a harmonic oscillator is a sinusoid with a given amplitude, frequency, and phase:



Suppose we have forgotten how the constants in the solution relate to the physical parameters of the oscillator. Let's plug in the proposed solution and look at the residual:



The residual here shows that for nonzero amplitude, the only solutions allowed are ones where ( k - m 2 ) = 0 or = (k/m)1/2.

**Exercise 1.11.**

Compute Lagrange's equations for the Lagrangians in exercise [1.9](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html#%_thm_1.9) using the Lagrange-equations procedure. Additionally, use the computer to perform each of the steps in the Lagrange-equations procedure and show the intermediate results. Relate these steps to the ones you showed in the hand derivation of exercise [1.9](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html#%_thm_1.9).

**Exercise 1.12.**

**a**.  Write a procedure to compute the Lagrange equations for Lagrangians that depend upon acceleration, as in exercise [1.10](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html#%_thm_1.10). Note that Gamma can take an optional argument giving the length of the initial segment of the local tuple needed. The default length is 3, giving components of the local tuple up to and including the velocities.

**b**.  Use your procedure to compute the Lagrange equations for the Lagrangian



Do you recognize the resulting equation of motion?

**c**.  For more fun, write the general Lagrange equation procedure that takes a Lagrangian of any order, and the order, to produce the required equations of motion.

This result was initially discovered by Euler and later rederived by Lagrange.

The derivative or partial derivative of a function that takes structured arguments is a new function that takes the same number and type of arguments. The range of this new function is itself a structure with the same number of components as the argument with respect to which the function is differentiated.

Lagrange's equations are traditionally written in the form



or, if we write a separate equation for each component of q, as



In this way of writing Lagrange's equations the notation does not distinguish between L, which is a real-valued function of three variables (t, q, ), and L o [q], which is a real-valued function of one real variable t. If we do not realize this notational pun, the equations don't make sense as written -- L/ is a function of three variables, so we must regard the arguments q, as functions of t before taking d/dt of the expression. Similarly, L/ q is a function of three variables, which we must view as a function of t before setting it equal to d/dt( L/ ). These implicit applications of the chain rule pose no problem in performing hand computations -- once you understand what the equations represent.

The variation operator is like the derivative operator in that it acts on the immediately following function:

f[q] = ( f)[q].

A function of multiple arguments is considered a function of a tuple of its arguments. Thus, the derivative of a function of multiple arguments is a tuple of the partial derivatives of that function with respect to each of the arguments. So in the case of a Lagrangian L,



To make this argument more precise requires careful analysis.

When we write a definition that names the components of the local tuple, we indicate that these are grouped into time, position, and velocity components by separating the groups with semicolons.

The derivative with respect to a tuple is a tuple of the partial derivatives with respect to each component of the tuple (see the appendix on notation).

The symbol is just a mnemonic symbol; the dot over the does not indicate differentiation. To define L we could have just as well have written:

L(a, b, c) = (1/2) m l2 c2 + m g l cos b.

However, we use a dotted symbol to remind us that the argument matching a formal parameter, such as , is a rate of change of an angle, such as .

In traditional notation these equations read



There is a Lagrange equation for every degree of freedom. The residuals of all the equations are zero if the path is realizable. The residuals are arranged in a down tuple because they result from derivatives of the Lagrangian with respect to argument slots that take up tuples. See the appendix on notation.

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**Lecture 3**

#### Lecture topic: [How to Find Lagrangians](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.6). [Hamilton's principle](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_Temp_94)

**The plan**

#### [1. How to Find Lagrangians](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.6)

#### 2. [Hamilton's principle](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_Temp_94)

[3. Coordinate Transformations](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.6.1)

#### [1. How to Find Lagrangians](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.6)

Lagrange's equations are a system of second-order differential equations. In order to use them to compute the evolution of a mechanical system, we must find a suitable Lagrangian for the system. There is no general way to construct a Lagrangian for every system, but there is an important class of systems for which we can identify Lagrangians in a straightforward way in terms of kinetic and potential energy. The key idea is to construct a Lagrangian L such that Lagrange's equations are Newton's equations = m.

Suppose our system consists of N particles indexed by , with mass m and vector position (t). Suppose further that the forces acting on the particles can be written in terms of a gradient of a potential energy that is a function of the positions of the particles and possibly time, but does not depend on the velocities. In other words, the force on particle is

= - ,

where is the gradient of with respect to the position of the particle with index . We can write Newton's equations as



Vectors can be represented as tuples of components of the vectors on a rectangular basis. So 1(t) is represented as the tuple **x**1(t). Let V be the potential energy function expressed in terms of components:



Newton's equations are



where 1, V is the partial derivative of V with respect to the **x**(t) argument slot.

To form the Lagrange equations we collect all the position components of all the particles into one tuple x(t), so

x(t) = (**x**0(t), ..., **x**N-1(t)).

The Lagrange equations for the coordinate path x are



Observe that Newton's equations ([1.51](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-13.html#EQUATION_1.51)) are just the components of the Lagrange equations ([1.54](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-13.html#EQUATION_1.54)) if we choose L to have the properties



here V(t, x(t)) = V(t; **x**0(t), ..., **x**N-1(t)) and 1, V(t, x(t)) is the tuple of the components of the derivative of V with respect to the coordinates of the particle with index , evaluated at time t and coordinates x(t). These conditions are satisfied if for every **a** and **b**



and



where a = (**a**0, ... , **a**N-1). We use the symbols a and b to emphasize that these are just formal parameters of the Lagrangian. One choice for L that has the required properties ([1.56](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-13.html#EQUATION_1.56)-[1.57](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-13.html#EQUATION_1.57)) is



where v2 is the sum of the squares of the components of **v**.

The first term is the kinetic energy, conventionally denoted T. So this choice for the Lagrangian is L(t, x, v) = T(t, x, v) - V(t, x), the difference of the kinetic and potential energy. We will often extend the arguments of the potential energy function to include the velocities so that we can write L = T - V.6

####

#### 2. [Hamilton's principle](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_Temp_94)

Given a system of point particles for which we can identify the force as the (negative) derivative of a potential energy V that is independent of velocity, we have shown that the system evolves along a path that satisfies Lagrange's equations with

L = T - V.

Having identified a Lagrangian for this class of systems, we can restate the principle of stationary action in terms of energies. This statement is known as Hamilton's principle: A point-particle system for which the force is derived from a velocity-independent potential energy evolves along a path q for which the action



is stationary with respect to variations of the path q that leave the endpoints fixed, where L = T - V is the difference between kinetic and potential energy.[63](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-13.html%22%20%5Cl%20%22footnote_Temp_95)

It might seem that we have reduced Lagrange's equations to nothing more than = m , and indeed, the principle is motivated by comparing the two equations for this special class of systems. However, the Lagrangian formulation of the equations of motion has an important advantage over = m . Our derivation used the rectangular components **x** of the positions of the constituent particles for the generalized coordinates, but if the system's path satisfies Lagrange's equations in some particular coordinate system, it must satisfy the equations in any coordinate system. Thus we see that L = T - V is suitable as a Lagrangian with any set of generalized coordinates. The equations of variational mechanics are derived the same way in any configuration space and any coordinate system. In contrast, the Newtonian formulation is based on elementary geometry: In order for D2(t) to be meaningful as an acceleration, (t) must be a vector in physical space. Lagrange's equations have no such restriction on the meaning of the coordinate q. The generalized coordinates can be any parameters that conveniently describe the configurations of the system.

####

#### [Constant acceleration](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_Temp_96)

Consider a particle of mass m in a uniform gravitational field with acceleration g. The potential energy is m g h where h is the height of the particle. The kinetic energy is just (1/2) mv2. A Lagrangian for the system is the difference of the kinetic and potential energies. In rectangular coordinates, with y measuring the vertical position and x measuring the horizontal position, the Lagrangian is

L(t; x, y; vx, vy) = (1/2) m ( vx2 + vy2 ) - m g y.

We have



This equation describes unaccelerated motion in the horizontal direction

(mD2 x(t) = 0) and constant acceleration in the vertical direction (mD2 y(t) = - g m).

####

#### [Central force field](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_Temp_98)

Consider planar motion of a particle of mass m in a central force field, with an arbitrary potential energy U(r) depending only upon the distance r to the center of attraction. We will derive the Lagrange equations for this system in both rectangular coordinates and polar coordinates.

In rectangular coordinates (x, y), with origin at the center of attraction, the potential energy is

V(t; x, y) = U((x2 + y2)1/2)

and the kinetic energy is

T(t; x, y; vx, vy) = (1/2) m (vx2 + vy2).

A Lagrangian for the system is L = T - V:



As a procedure:



We can rewrite these Lagrange equations as:



where

r(t) = ((x(t))2 + (y(t))2)1/2.

We can interpret these as follows. The particle is subject to a radially directed force with magnitude - DU(r). Newton's equations equate the force with the product of the mass and the acceleration. The two Lagrange equations are just the rectangular components of Newton's equations.

We can describe the same system in polar coordinates. The relationship between rectangular coordinates (x, y) and polar coordinates (r, ) is



The relationship of the generalized velocities is derived from the coordinate transformation. Consider a configuration path that is represented in both rectangular and polar coordinates. Let and be components of the rectangular coordinate path, and let and be components of the corresponding polar coordinate path. The rectangular components at time t are ((t), (t)) and the polar coordinates at time t are ((t), (t)). They are related by ([1.62](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-13.html#EQUATION_1.62)):



The rectangular velocity at time t is (D(t), D(t)). Differentiating ([1.63](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-13.html#EQUATION_1.63)) gives the relationship among the velocities



These relations are valid for any configuration path at any moment, so we can abstract them to relations among coordinate representations of an arbitrary velocity. Let vx and vy be the rectangular components of the velocity and and be the rate of change of r and . Then



The kinetic energy is (1/2) m(vx2 + vy2):



and the Lagrangian is



We express this Lagrangian as follows:

Lagrange's equations are



We can interpret the first equation as saying that the product of the mass and the radial acceleration is the sum of the force due to the potential and the centrifugal force. The second equation can be interpreted as saying that the derivative of the angular momentum mr2D is zero, so angular momentum is conserved.

Note that we used the same Lagrange-equations procedure for the derivation in both coordinate systems. Coordinate representations of the Lagrangian are different for different coordinate systems, and the Lagrange equations in different coordinate systems look different. Yet the same method is used to derive the Lagrange equations in any coordinate system.

**Exercise 1.13.**

Check that the Lagrange equations for central force motion in polar coordinates and in rectangular coordinates are equivalent. Determine the relationship among the second derivatives by substituting paths into the transformation equations and computing derivatives, then substitute these relations into the equations of motion.

###

### [3. Coordinate Transformations](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.6.1)

The motion of a system is independent of the coordinates we use to describe it. This coordinate-free nature of the motion is apparent in the action principle. The action depends only on the value of the Lagrangian along the path and not on the particular coordinates used in the representation of the Lagrangian. We can use this property to find a Lagrangian in one coordinate system in terms of a Lagrangian in another coordinate system.

Suppose we have a mechanical system whose motion is described by a Lagrangian L that depends on time, coordinates, and velocities. And suppose we have a coordinate transformation F such that

x = F(t, x').

The Lagrangian L is expressed in terms of the unprimed coordinates. We want to find a Lagrangian L' expressed in the primed coordinates that describes the same system. One way to do this is to require that the value of the Lagrangian along any configuration path be independent of the coordinate system. If q is a path in the unprimed coordinates and q' is the corresponding path in primed coordinates, then the Lagrangians must satisfy:



We have seen that the transformation from rectangular to polar coordinates implies that the generalized velocities transform in a certain way. The velocity transformation can be deduced from the requirement that a path in polar coordinates and a corresponding path in rectangular coordinates are consistent with the coordinate transformation. In general, the requirement that paths in two different coordinate systems be consistent with the coordinate transformation can be used to deduce how all of the components of the local tuple transform. Given a coordinate transformation F, let C be the corresponding function that maps local tuples in the primed coordinate system to corresponding local tuples in the unprimed coordinate system:



We will deduce the general form of C below.

Given such a local-tuple transformation C, a Lagrangian L' that satisfies equation ([1.68](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-13.html#EQUATION_1.68)) is



We can see this by substituting for L' in equation ([1.68](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-13.html#EQUATION_1.68)):



To find the local-tuple transformation C given a coordinate transformation F, we deduce how each component of the local tuple transforms. Of course, the coordinate transformation specifies how the coordinate component of the local tuple transforms. The generalized-velocity component of the local-tuple transformation can be deduced as follows. Let q and q' be the same configuration path expressed in the two coordinate systems. Substituting these paths into the coordinate transformation and computing the derivative, we find



Through any point there is always a path of any given velocity, so we may generalize and conclude that along corresponding coordinate paths the generalized velocities satisfy



If needed, rules for higher-derivative components of the local tuple can be determined in a similar fashion. The local-tuple transformation that takes a local tuple in the primed system to a local tuple in the unprimed system is constructed from the component transformations:



So if we take the Lagrangian L' to be



then the action has a value that is independent of the coordinate system used to compute it. The configuration path of stationary action does not depend on which coordinate system is used to describe the path. The Lagrange equations derived from these Lagrangians will in general look very different from one another, but they must be equivalent.

**Exercise 1.14.**

Show by direct calculation that the Lagrange equations for L' are satisfied if the Lagrange equations for L are satisfied.

Given a coordinate transformation F, we can use ([1.74](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-13.html#EQUATION_1.74)) to find the function C that transforms local tuples. The procedure F->C implements this:[65](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-13.html%22%20%5Cl%20%22footnote_Temp_101)

As an illustration, consider the transformation from polar to rectangular coordinates, x = r cos and y = r sin , with the following implementation:

In terms of the polar coordinates and the rates of change of the polar coordinates, the rates of change of the rectangular components are



We can use F->C to find the Lagrangian for central force motion in polar coordinates from the Lagrangian in rectangular components, using equation ([1.70](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-13.html#EQUATION_1.70)):



The result is the same as Lagrangian ([1.67](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-13.html#EQUATION_1.67)).

**Exercise 1.15.**  **Central force motion**

Find Lagrangians for central force motion in three dimensions in rectangular coordinates and in spherical coordinates. First, find the Lagrangians analytically, then check the results with the computer by generalizing the programs that we have presented.

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**Lecture 4**

### Lecture topic: [Constrained Motion](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.10). [Nonholonomic Systems](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.10.3)

**The plan**

## [1. Constrained Motion](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.10)

### [2. Coordinate Constraints](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.10.1)

### [3. Derivative Constraints](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.10.2)

### [4. Nonholonomic Systems](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.10.3)

## [5. Summary](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.11) and Projects

## [1. Constrained Motion](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.10)

Suppose the configuration of a system with n degrees of freedom is specified by n + 1 coordinates and that configuration paths q are constrained to satisfy some relation of the form



The condition that the action is stationary still reduces to the condition ([1.34](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-12.html#EQUATION_1.34)):



Now the choice of is not completely free. So we have



with  subject to the constraints.

A path q satisfies the constraint if [q] = o [q] = 0. The constraint must be satisfied even for the varied path, so we allow only variations for which the variation of the constraint is zero:



We can say that the variation must be ``tangent'' to the constraint surface. Expanding this with the chain rule, a variation is tangent to the constraint surface if



Note that these are functions of time; the variation at a given time is tangent to the constraint at that time.

###

### [2. Coordinate Constraints](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.10.1)

Consider constraints that do not depend on velocities:



In this case the variation is tangent to the constraint surface if



Thus, the residual of Lagrange's equations is parallel to the normal to the constraint surface; the two must be proportional:



These equations, with the constraint equation o [q] = 0, are the governing equations. These equations are sufficient to determine the path q and to eliminate the unknown function .

####

#### [Now watch this](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_Temp_157)

Suppose we form an augmented Lagrangian treating as one of the coordinates:



Notice that this Lagrangian is of the same form as the Lagrangian (equation [1.89](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-13.html#EQUATION_1.89)) that we used in the derivation of L = T - V for rigid systems.

#### [Alternatively](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html%22%20%5Cl%20%22%25_toc_%25_sec_Temp_158)

If could be written as a function of the solution state path, then it would be clear that it is determined by the state and can thus be eliminated. Suppose can be written as a composition of a state-dependent function with the path: = o [q]. Consider the Lagrangian



Applying the Euler-Lagrange operator E (see section [1.9](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-16.html#%_sec_1.9)) to this Lagrangian gives[90](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-17.html%22%20%5Cl%20%22footnote_Temp_159)



Composition of E[L''] with [q] gives the Lagrange equations for the path q. Using the fact that the constraint is satisfied on the path o [q] = 0 and consequently Dt o [q] = 0, we have



where we have used = o [q]. If we now use the fact that we are dealing only with coordinate constraints, 2 = 0, then



The difference is that now we see that = o [q] is determined by the unaugmented state. This is the same as saying that can be eliminated.



####

#### [The pendulum using constraints](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_Temp_160)

A Lagrangian for the unconstrained particle is



The constraint that the pendulum moves in a circle of radius l about the pivot is[91](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-17.html%22%20%5Cl%20%22footnote_Temp_161)



The augmented Lagrangian is



The Lagrange equations for the augmented Lagrangian are



It should not be surprising that these equations simplify if we switch to ``polar'' coordinates



Substituting this into the constraint equation, we determine that r = l, a constant. Forming the derivatives and substituting into the other two equations, we find



Multiplying the first by cos and the second by sin and adding, we find



which we recognize as the correct equation for the pendulum. This is the same as the Lagrange equation for the pendulum using the unconstrained generalized coordinate . For completeness, we can find in terms of the other variables:



####

#### [Building systems from parts](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_Temp_163)

Consider the compound spring-mass system shown at the top of figure [1.9](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-17.html#FIGURE_1.9). We could analyze this as a monolithic system with two configuration coordinates x1 and x2, representing the extensions of the springs from their equilibrium lengths X1 and X2.

We can then choose a Lagrangian for the composite system as the sum of the two component Lagrangians with a constraint = X1 + x1 to accomplish the coupling.



Let's see how this works. The Lagrangian for the subsystem attached to the wall is



and the Lagrangian for the subsystem that attaches to it is



We construct a Lagrangian for the system composed from these parts as a sum of the Lagrangians for each of the separate parts, with a coupling term to enforce the constraint:



Thus we can write Lagrange's equations for the four configuration coordinates, in order, as follows:



We can now eliminate the ``glue'' coordinates and to obtain the equations of motion in the coordinates x1 and x2:



This strategy can be generalized. We can make a library of primitive components.

**Exercise 1.34.**  **Combining Lagrangians**

**a**.  Make another primitive component, compatible with the spring-mass structures described in this section. For example, make a pendulum that can attach to the spring-mass system. Build a combination and derive the equations of motion. Be careful, the algebra is horrible if you choose bad coordinates.

**b**.  For a nice little project, construct a family of compatible mechanical parts, characterized by appropriate Lagrangians, that can be combined in a variety of ways to make interesting mechanisms. Remember that in a good language the result of combining pieces should be a piece of the same kind that can be further combined with other pieces.

**Exercise 1.35.**  **Motion of a tiny golf ball**

Consider the motion of a golf ball idealized as a point mass constrained to a frictionless smooth surface of varying height h(x, y) in a uniform gravitational field with acceleration g.

**a**.  Find an augmented Lagrangian for this system, and derive the equations governing the motion of the point mass in x and y.

**b**.  Under what conditions is this approximated by a potential function V(x, y) = mgh(x, y)?

**c**.  Assume that h(x, y) is axisymmetric about x = y = 0. Can you find such an h that yields motions with closed orbits?

###

### [3. Derivative Constraints](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.10.2)

Consider a velocity-dependent constraint = 0. That is a total time derivative means that there exists a velocity-independent function such that



That is velocity-independent means 2 = 0. As state functions the relationship between and is



Given a we can find by solving this linear partial differential equation. The solution is determined up to a constant, so = 0 implies = K for some constant K. On the other hand, if we knew = K then = 0 follows. Thus the velocity-dependent constraint = 0 is equivalent to the velocity-independent constraint = K, and we know how to find Lagrange equations for such systems.

If L is a Lagrangian for the unconstrained problem, the Lagrange equations with the constraint = K are



where is a function of time that will be eliminated during the solution process. The constant K does not affect the Lagrange equations. The function is independent of velocity, 2 = 0, so the Lagrange equations become



From equation ([1.208](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-17.html#EQUATION_1.208)) we see that



so the Lagrange equations with the constraint = 0 are



These Lagrange equations are given if we augment the Lagrangian with the constraint multiplied by a function of time ':



The Lagrange equations for L' turn out to be



which, with the identification = - D', are the same as Lagrange equations ([1.212](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-17.html#EQUATION_1.212)).

**Exercise 1.37.**

Show that the augmented Lagrangian ([1.213](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-17.html#EQUATION_1.213)) does lead to the Lagrange equations ([1.214](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-17.html#EQUATION_1.214)), taking into account the fact that is a total time derivative of .

####

#### [Goldstein's hoop](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_Temp_168)

Here we consider a problem for which the constraint can be represented as a time derivative of a coordinate constraint: a hoop of mass M rolling, without slipping, down a (one-dimensional) inclined plane (see figure [1.10](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-17.html#FIGURE_1.10)).



Thus a change in is exactly reflected in a change in x; the constraint function is



x = R + c.

The potential energy of the hoop decreases as the height decreases. Thus we may write the augmented Lagrangian:



Lagrange's equations are



And by differentiation of the third Lagrange equation we obtain









###

### [4. Nonholonomic Systems](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.10.3)

Systems with constraints that are not integrable are termed nonholonomic systems. An example of a nonholonomic system is a ball rolling without slipping in a bowl. As the ball rolls it must turn so that its surface does not move relative to the bowl at the point of contact.

Let have the form



a state function that is linear in the velocities. We assume is not a total time derivative. If L is a Lagrangian for the unconstrained system, then the equations of motion are asserted to be



An essential step in the derivation of the Lagrange equations for coordinate constraints = 0 with 2 = 0 was to note that two conditions must be satisfied:



and



Because E [L] o [q] is orthogonal to and is constrained to be orthogonal to 1 o [q] , the two must be parallel at each moment:



In this case, for a variation to be consistent with the velocity-dependent constraint function it must satisfy (see equation [1.179](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-17.html#1.179))



Define a ``virtual velocity'' to be any velocity satisfying



The ``principle of d'Alembert-Lagrange,'' according to Arnold, states that



for any virtual velocity . Because is arbitrary except that it is required to be orthogonal to 2 o [q] and any such is orthogonal to E [L] o [q], then 2 o [q] must be parallel to E [L] o [q]. So



which are the nonholonomic equations.

All ``derivations'' of the nonholonomic equations have similar identifications.



then the Lagrange equations associated with the coordinates are



The Lagrange equation associated with is just the constraint equation



Recall that the Euler-Lagrange operator E has the property



## [5. Summary](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.11) and Projects

To analyze a mechanical system we construct an action function that gives us a way to distinguish realizable motions from other conceivable motions of the system. The action function is constructed so as to be stationary only on paths describing realizable motions, with respect to variations of the path. This is called the principle of stationary action.

## [*Projects*](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_1.12)

**Exercise 1.37.**  **A numerical investigation**

Consider a pendulum: a mass m supported on a massless rod of length l in a uniform gravitational field. A Lagrangian for the pendulum is



Consider the parameters m = 1 kg, l = 1 m, g = 9.8 m s-2. The frequency of small-amplitude oscillations is 0 = (g/l)1/2. Let's find the nontrivial solution that has the frequency 1 = (4/5) 0.

**a**.  The angle is periodic in time, so a Fourier series representation is appropriate. Since the potential is even in the angle, the angle is an odd function of time. Thus only odd terms of the series are present:



The amplitude of the trajectory is A = max = sumn=1infty ( - 1)n+1 An.

Find approximations to the first few coefficients An by minimizing the action.

**b**.  Now let's formulate the analytic solution for the frequency as a function of amplitude. The period of the motion is simply



We still have the problem that we can specify the amplitude A and get the frequency; to solve our problem we need to solve the inverse problem.

**Exercise 1.38.**  **Double pendulum behavior**

**a**.  Formulate a Lagrangian to describe the dynamics. Derive the equations of motion in terms of the given angles 1 and 2. Put the equations into a form appropriate for numerical integration. Assume the following system parameters:



**b**.  Prepare graphs showing the behavior of each angle as a function of time when the system is started with the following initial conditions:



Make the graphs extend to 50 seconds. Save the state points at .125-second intervals in a list.

**c**. Repeat the previous comparison, but this time with the initial conditions:



What do you see here?



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**Lecture 5**

**Lecture topic: Elastic Strain Energy for Various Types of Loading**

**The plan**

**1. External Work**

1.1. Work of a Force

1.2. Work of a Couple

**2. Strain Energy**

2.1. Normal Stress

2.2. Shear Stress

2.3. Multi-axial Stress

**3. Elastic Strain Energy for Various Types of Loading**

3.1. Axial Load

3.2. Bending Moment

3.3. Transverse Shear

3.4. Torsional Moment

**4. Conservation of Energy**

4.1. Trusses

4.2. Vertically Loaded Beams

4.3. Beams Loaded with a Couple

**5. Impact Loading**

**1. External Work**

***1.1. Work of a Force***

* The work done by a force is equivalent to the product of the component of the force acting in the direction of motion and the distance travelled.



* If the force acts in the -direction:



* If a force is applied to a prismatic beam in a gradual manner, i.e. the magnitude of the force increases from 0 to , and the bar stretches by , when the material behaves in a linear-elastic manner  then:

 (1)

***1.2. Work of a Couple***

* A couple moment does work as it goes through a rotation:



* If a moment is applied to a body with linear-elastic material behavior such that the magnitude of the couple increases from 0  to   then:

 (2)

**2. Strain Energy**

* External work done by loads applied to a body will be converted into strain energy. This strain energy is cause by normal and shear stresses that deform the body.

***2.1. Normal Stress***

* Consider a body deformed by a normal stress EQS:
* The force on the top face is  and if it is applied gradually as the element undergoes deformation  the work done by the force is (using Eq. (1)):



* Or



* So if a body is subjected to uni-axial normal stress, the strain energy is:

 (3)

* For linear-elastic material behavior, Hooke’s Law  applies, and:

 (4)

* Note:  is always positive.

***2.2. Shear Stress***

* Consider an element subjected to shear stress, :
* The force  on the top face will move . Assuming  is applied gradually, and using Eq. (1):



* Or

 (5)

* For linear-elastic behavior, Hooke’s Law  applies, and:

 (6)

***2.3. Multi-axial Stress***

* Consider an element subjected to a general state of stress.
* Assuming linear-elastic behavior and all loads are applied gradually, the strain energy associated with each normal and shear stress can be added to give:

 (7)

* Using the generalized Hooke’s Law:







 ;  ; 

* The strains can be eliminated from Eq. (6-7):

 (8)

* And if only the principal stresses act on the element (i.e. ,  and )

 (9)

**3. Elastic Strain Energy for Various Types of Loading**

***3.1. Axial Load***

* Consider a bar with a slowly changing cross-section that is loaded centroidally.
* The internal load at  from one end is , and the normal stress is . using Eq. (4) the strain energy is:



* The volume can be expressed as  and:

 (10)

* If the cross-sectional area is constant:

 (11)

* Note:
	+ , 
	+ , 
	+ , 
	+ i.e. something that is easy to distort will store more strain energy.

***3.2. Bending Moment***

* Application of a bending moment to a straight prismatic member results in a normal stress.
* Consider the element of area ,  from the neutral axis, then , and using Eq. (4):



* The volume  can be written as , so:



* Will give the strain energy in the member, and since

:

 (12)

* Note: The bending moment needs to be expressed as a function of , then Eq. (12) can be integrated.

***3.3. Transverse Shear***

* Consider a prismatic beam with an axis of symmetry .
* The internal shear force at  is , and the shear stress on the element of area  is . Using Eq. (6) the strain energy is:



* Or

 (13)

* Defining the form factor, , which is a function of geometry:

 (14)

* The strain energy can be written as;

 (15)

* An example of the form factor calculation is given in the text. For a rectangular cross-section .
* Note:  due to shear is usually much less than  for bending (se e.g.14.4, Hibbeler, 6e) and the shear strain energy stored in beams is usually neglected.

***3.4. Torsional Moment***

* Consider a shaft with a gradually changing cross-section:
* If the shaft is subjected to an internal torque  at  from one end, the shear stress on the element  at  from the centroid is , and using Eq. (6) the strain energy is;



* Or

 (16)

* But the polar moment of inertia, J, is defined as:

 (17)

* Using Eq. (17) the strain energy can be written:

 (18)

* If the shaft (or tube) has constant cross-sectional area:

 (19)

**4. Conservation of Energy**

* The principal of conservation of energy states: Energy is a conserved property. It can neither be created nor destroyed; only its form can be altered from one form of energy to another.
* Only mechanical energy will be considered, but kinetic energy will be neglected since all loadings will be gradual.
* Conservation of energy would require that the external work done by applied loads (i.e. applied loads that cause deflections) must be equivalent to the strain energy developed in a body as it deforms.

 (20)

* If the loads are removed the stored strain energy will restore the body to its undeformed state (if the elastic limit has not been exceeded).

***4.1. Trusses***

* Consider a truss subjected to the load .
* If the point of application of the load  deflects  in the direction of , and the load is increased gradually from 0 to , then from Eq. (1):

 (21)

* This external work done on the body is stored as strain energy. If, due to , the axial force  develops in a member, the strain energy stored in that member is  from Eq. (11). To determine the total strain energy stored in the truss:

 (22)

* Where the summation is over all the members in the truss.
* Conservation of energy requires , therefore:

 (23)

* The deflection  caused by  can be evaluated after the axial forces in each member of the truss has been determined using statics.

***4.2. Vertically Loaded Beams***

* Consider a beam loaded with the vertical force P.
* The deflection at the point of application of P can be determined from the conservation of energy, Eq. (20), using Eqs. (1) and (12), for  and , respectively:

 (24)

* The bending moment would be written as a function of .
* ***Note:*** the beam deflects due to bending moment and shear, however, the strain energy due to shear is usually neglected, thus the deflection can be written as a function of bending moment only.

***4.3. Beams Loaded with a Couple***

* Consider a cantilever beam subjected to an applied moment .
* The couple moment will cause the rotation  at the point of application, and it does work due to this rotation:  from Eq. (2).
* The strain energy would be caused by the bending moment , and



from Eq. (12).

* Conservation of energy, Eq. (20), would require:

 (25)

* Where  is a function of .
* Note: Application of the conservation of energy is limited to situations where only one applied load exists. For multiple applied loads, each load would have an associated external work and deflection, but there is only one conservation equation, so only one unknown deflection can be solved.

**5. Impact Loading**

* Remember Mechanics II? Remember work-energy and conservation of energy methods?
* E.g. A weight is dropped from rest from a height  on to a linear spring, with spring constant . What is the maximum deflection of the spring?
* Conservation of energy:





* Or Work-Energy:  strain energy in the spring.



* The result can be rearranged to give:



* The quadratic equation can be solved to give the maximum root:

 (26)

* If the weight is applied statically (i.e. gradually)  or , and Eq. (6-26) can be written as:



* Or

 (27)

* Where the term in the square root is the extra displacement due to dynamic loading.
* Note: if , i.e. the weight W is released while it just touches the spring, .
* E.g. A weight *W* travelling with velocity  on a frictionless horizontal surface impacts a linear spring, with spring constant . What is the maximum deflection of the spring?
	+ Conservation of Energy;





* + Or

 (28)

* + A statically loaded spring would deflect , so Eq. (6-28) can be written as:

 (29)

* How to convert this information into deflections of dynamically loaded members? I.e., how is impact loading simulated?
* Assume:
	1. The moving body is rigid.
	2. The stationary body deforms in a linear-elastic manner (i.e. it behaves as a linear spring).
	3. No energy is lost during the collision.
	4. The bodies remain in contact during the collision.
* These are conservative assumptions, which lead to overestimates of forces (i.e. good for design purposes).
* With these assumptions, the deformable body behaves like a linear spring.
* i.e. an effective spring constant can be defined and Eqs. (27) or (29) can be used to determine .
* An equivalent spring constant is not required. All that is needed is the static deflection, , for use in Eq. (27).  can be obtained from the equation of the elastic curve, Hooke’s Law, Appendix C, or conservation of energy and strain energy.
* An impact factor, , can be defined from Eq. (27):

. (30)

* So:



* And



* And the maximum stress is then: .

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**Lecture 6**

**Lecture topic: Internal Strain Energy Stored and External Work done**

**The plan**

**1. External Work done**

1.1. Due to an Axial Load on a Bar

1.2. Work done due to an end moment

1.3. Work done due to the externally applied torque T1

**2. Internal Energy Stored (or Internal Work Done)**

2.1. Due to an end axial force

2.2. Due to Shear Stresses and Strains

2.3. Due to a bending moment

2.4. Due to an axial force

2.5. Due to a transverse shear force

2.6. Due to a torsional moment

2.7. Due to Three dimensional Stresses and Strains

**1. External Work done**

***1.1. Due to an Axial Load on a Bar***

Consider a bar, of length L and cross-sectional area A, to be subjected to an end axial load P. Let the deformation of end B be Δ1. When the bar is deformed by axial load, it tends to store energy internally throughout its volume. The externally applied load P, acting on the bar, does work on the bar dependent on the displacement Δ1 at its end B, where the load is applied. Let this external work done by the load be designated as ue.

Drawing the force-deformation diagram of the bar, as it is loaded by P.





Since the force versus the end displacement relationship is linear, F at any displacement Δ can be represented by

F = k Δ, where k = a constant of proportionality

 (A)

The external work done on the bar by P increases from zero to the maximum as the load P increases from 0 to P (in a linear manner). Therefore the total work done can be represented by the average magnitude of externally applied force (viz., P/2), multiplied by the total displacement Δ1 (as given by equation (A)).

Let an additional load P′ be applied to the bar after the load P has caused an end extension of Δ1 at B. Considering the deformation of the end B of the bar due to the application of an additional load P′ at B, let the additional deformation of the bar be equal to Δ′.

Area = ½ PΔ1

The total external work done





Incremental work done on the bar if the load P′ is applied to the bar resulting in a displacement Δ′

Incremental work done on the bar when load P was applied at B, initially

Considering ΔS OEF and OED, Area of Figure GHIF = Area of Figure CDJH

i.e.,





Additional work done by P as the bar deforms by an additional Δ′

Hence when a bar (having a load P acting on it) is subjected to an additional load P′, then the work done by (the already acting) P due to the incremental deformation Δ′ (caused by P′) is equal to PΔ′. This is similar, to a suddenly applied load P creating an instantaneous deformation Δ, producing an external work of PΔ′.

***1.2. Work done due to an end moment***



Let a moment M be applied to end B of the beam AB. Let the rotation at end B be θ1 due to M. Since M and θ gradually increase from zero to θ1 (following earlier formulations for an axially loaded bar)





***1.3. Work done due to the externally applied torque T1***







**2. Internal Energy Stored (or Internal Work Done)**

***2.1. Due to an end axial force***

The internal strain energy stored in the material is dependent on the amount of stresses and strains created within the volume of the structure.



The internal strain energy Ui stored within the body is given by



 

***2.2. Due to Shear Stresses and Strains***

y







[Force on other faces do not do any work since motion of face ABCD is zero]



***2.3. Due to a bending moment***



 “I” - can be constant or varying

***2.4. Due to an axial force***





***2.5. Due to a transverse shear force***





***2.6. Due to a torsional moment***





***2.7. Due to Three dimensional Stresses and Strains***



***Multi-axial Stresses:*** The previous development may be expanded to determine the strain energy in a body when it is subjected to a general state of stress, Figure shown above. The strain energies associated with each of the normal and shear stress components can be obtained from Eqs. I and II. Since energy is a scalar, the strain energy in the body is therefore

  (I)

The strains can be eliminated by using the generalized form of Hook’s law given by Equations. After substituting and combining terms, we have

 (II)

where 

If only the principal stresses  act on the element, as shown in the earlier figure, this equation reduces to a simpler form, namely,

 (14)

Recall that we used this equation in Sec. 1 as a basis for developing the maximum-distortion-energy theory.

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**Lecture 7**

**Lecture topic: Impact Problems. Principle of Virtual Work.** **Specific Structures**

**The plan**

**1. Conservation of Energy**

**2. Impact Problems**

2.1. Impact Problems Using Energy Methods

2.2. Axial Impact of an Elastic Rod

2.3. Impact Response of an elastic spring

2.4. Impact Bending of a Beam

**3. Principle of Virtual Work. Virtual Work Method for Deflections (or Deformations)**

**4. Specific Structures**

4.1. Trusses

4.2. Beams

4.3. Consider a truss subjected to loads F1, F2 and F3

4.4. Considering a Beam Subjected to Bending Loads P1, P2 and P3

**5. Castigliano’s Theorem** (**Based on the strain energy stored in a body)**

**1. Conservation of Energy**

***Using the principle of conservation of energy***

*Internal strain energy stored in the structure due to the applied load = External work done by the applied load.*

***Appendix to: Effect of Transverse Shear Forces***

  (1)

To simplify this expression for Ur , let us define a new cross-sectional properties fs, called the form factor for shear. Let

  (2)

(The form factor is a dimensionless number that depends only on the shape of the cross section, so it rarely actually varies with x). Combining Eqs. 1 and 2 we get the following expression for the ***strain energy due to shear in bending****:*

  (3)

The form factor for shear must be evaluated for each shape of cross section. For example, for a rectangular cross section of width b and height h, the expression



was obtained in example Problem 6.14 (Chapter 6). Therefore, from Eq. 2 we get

  (4)

The form factor for other cross-sectional shapes is determined in a similar manner. Several of these are listed in Table A, given below. The approximation for an I-section or box section is based on assuming that the shear force is uniformly distributed over the depth of the web(s).

**Table A:** Form Factor fs for shear



**2. Impact Problems**

***2.1. Impact Problems Using Energy Methods***

What are impact forces?

 Suddenly applied forces that act for a short duration of time

* Collision of an automobile with a guard rail
* Collision of a pile hammer with the pile
* Dropping of a weight on to a floor



Loaded member vibrates till equilibrium is established.

***Assumptions:***

1. At impact, all kinetic energy of striking mass is entirely transferred to the structure. It is transferred as strain energy within the deformable body.



This means that the striking mass should not bounce off the structure and retain some of its kinetic energy.

1. No energy is lost in the form of heat, sound or permanent deformation of the striking mass.

***2.2. Axial Impact of an Elastic Rod***

δ



*vi* = velocity of impact



Equating

*Ui = Ue*





***2.3. Impact Response of an elastic spring***



Static deflection of spring 

 k = spring constant = load per unit deformation

  = maximum deflection of spring due to impact = δ

 Fe = maximum force in spring during impact 



If we use the velocity at impact as a parameter, just before impact



Substituting in Eqn. (I),

 (III)

***2.4. Impact Bending of a Beam***





For a central load,





Let





To find the impact bending stress,



**3. Principle of Virtual Work. Virtual Work Method for Deflections (or Deformations)**

Work-energy method, of equating the external work to internal strain energy, has the disadvantage that normally only the deflection (or deformation) caused by a single force can be obtained. The method of virtual work provides a general procedure to determine the deflections and slopes (or rotations) at any point in the structure (which can be a truss, a beam or frame) subjected a number of loadings.

To develop the virtual work method in a general manner, let us consider a body or a structure of arbitrary shape (later this body will be made to represent a specific truss, beam or frame) shown in the figure below.



Δ = Deformation at A, along AB, caused by the loads P1, P2 and P3.

Let us assume that we want to determine the deflection Δ of a point A, along the line AB, caused by a number of actual (or real) forces P1, P2 and P3 acting on the body, as shown in Figure (b). To find Δ at A, along AB, due to the applied loads (P1, P2 and P3), using the virtual work method, the following procedure could be used.



Figure (a)

**Step 1:** Place a virtual force (here we use a unit virtual force) on the body at point A in the same direction AB, along which the deflection is to be found. The term virtual force is used to indicate that the force is an imaginary one and does not exist as part of the real forces. This unit force, however, causes internal virtual forces throughout the body. A typical virtual force (acting on a representative element of the body) is shown in Figure (a).



Figure b

**Step 2:** Next place the real forces, P1, P2 and P3 on the body [Figure (b)]. These forces cause the point A to deform by an amount Δ along the line AB, while the representative element, of length L, now deforms by an amount dL. As these deformations occur within the body, the external unit virtual force (already acting on the body before P1, P2 and P3 are applied) moves through the displacement Δ; similarly the internal virtual force u acting on the element (before P1, P2 and P3 are applied) moves through the displacement dL. These forces, moving through displacements Δ and dL, do work.

**Step 3:** The external virtual unit force, moving through displacement Δ, performs external virtual work given as (1) times (Δ), on the body. Similarly, the internal virtual force u, moving through displacement dL, performs internal virtual work given as (u) times (dL). Since the external virtual work is equal to the internal virtual work done on all elements making up the body, we express the virtual work equation as:



The summation sign, in Eqn. (A), indicates that all the internal virtual work in the whole body must be included. Eqn. (A) gives the deflection Δ along the line of action of unit virtual force. A positive value for Δ indicates that the deflection is in the same direction as the unit force.

In writing down Eqn. (A), one has to remember that the full values of the virtual forces (unit force at A, and all the internal forces, ui) were already acting on the body when the real forces were applied (viz. P1, P2 and P3). Therefore, no one-half appears in any term of Eqn. (A).

In a similar manner, the rotation (or slope) at a point in a body can be determined by applying a virtual unit moment or couple (instead of a unit force) at the point where the rotation is desired (see Figure below).



**(a) Virtual unit moment applied** **(b) Real forces P1, P2 and P3 applied**

(Develop virtual force u, within (Virtual unit moment rotates through an the body) angle θ)



**4. Specific Structures**

***4.1. Trusses***

1. **Subjected to applied external loads only**

If ui represents the internal forces developed in the members, due to an applied unit load (at the point where the deformation is to obtained) in the required direction, then Eqn. (A) can be expressed as

  (C)

1. **For trusses subjected to a temperature change (causing internal forces)**

The incremental deformation caused in member due to a temperature rise is dL, where



Also



1. **Trusses with Fabrication Errors**

 (E)

where ΔL = difference in length of the member from its intended length, caused by a fabrication error.

***4.2. Beams***

For loads acting on a beam subjected to bending moments alone, the deformation Δ, at a given point along a given direction is given by

  (F)

where *m* is the bending moment in the member when a unit load is applied on the structure at the specified point in the specified direction. For a general loading on the beam, generating axial, shear, bending and torsional forces/moments in the beam

  (G)

where *n* is the axial force generated in the beam when a unit load is applied on the beam in the required direction; similarly m, v and t are the bending moment, shear force and torsional moment generated under the applied unit load.

***4.3. Consider a truss subjected to loads F1, F2 and F3***



Unit virtual load is applied in the direction in which the deflection is required, say at B in the vertical direction. Let uAB, uBC, uCA and uCD be the internal forces generated when the unit load is applied at B.



Let PAB, PBC, PCA and PCD be the internal forces generated in the truss members due to the given loads F1, F2 and F3 acting on the beam. Then the vertical deflection at B is obtained as,

  (H)

***4.4. Considering a Beam Subjected to Bending Loads P1, P2 and P3***



Let us say that it is required to find the vertical deflection at C due to the given loads.

1. Apply a unit vertical load (virtual) at C in the vertical direction and find the moment m in the beam.
2. Then apply the given loads on the beam (say P1, P2 and P3) and compute the bending moments M in the beam. Then the deflection Δv at C is obtained



 (I)

**5. Castigliano’s Theorem** (**Based on the strain energy stored in a body)**

Consider a beam AB subjected to loads P1 and P2, acting at points B1 and B2 , respectively.





If ,

 where f11 = deflection at B1 due to a unit load at B1

 and  with f21 = deflection at B2 due to a unit load at B1

and

 , with f22 = deflection at B2 due to a unit load at B2 &

, with f12 = deflection at B1 due to a unit load at B2.

Then



Similarly,



Considering the work done = Ui

 

Now we reverse the order the application of loads P1 and P2, viz., applying P2 at B2 first and then applying P1 at B1,





Similarly,



Ui =

 

Considering equation (III) and (IV), and equating them, it can be shown that





This is called Betti – Maxwell’s reciprocal theorem



Deflection at B2 due to a unit load at P1 is equal to the deflection at B1 due to a unit load at P2.

From Eqn. (III)



From Eqn. (IV)



This is Castigliano’s first theorem.

Similarly the energy Ui can be express in terms of spring stiffnesses k11, k12 (or k21), & k22 and deflections v1 and v2; then it can be shown that



This is Castigliano’s second theorem. When rotations are to be determined, 

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**Lecture 8**

**Lecture topic: Method of Virtual Work. Analysis of Frames and Trusses**

**The plan**

1. Energy principles

2. Computations of Deflections (deformations: Beams; Frames and Trusses)

3. The Method of Virtual Work and Energy

4. Analysis of Frames and Trusses

**1. Energy principles**

Energy principles in structural mechanics express the relationships between stresses, strains or deformations, displacements, material properties, and external effects in the form of energy or work done by internal and external forces. Since energy is a scalar quantity, these relationships provide convenient and alternative means for formulating the governing equations of deformable bodies in solid mechanics. They can also be used for obtaining approximate solutions of fairly complex systems, bypassing the difficult task of solving the set of governing partial differential equations.

**General principles**

***Virtual work*** arises in the application of the ***principle*** of least action to the study of forces and movement of a mechanical system. The work of a force acting on a particle as it moves along a displacement will be different for different displacements. Among all the possible displacements that a particle may follow, called virtual displacements, one will minimize the action. This displacement is therefore the displacement followed by the particle according to the principle of least action. The work of a force on a particle along a virtual displacement is known as the virtual work.

Historically, virtual work and the associated calculus of variations were formulated to analyze systems of rigid bodies,[1] but they have also been developed for the study of the mechanics of deformable bodies.

***The principle of virtual work*** had always been used in some form since antiquity in the study of statics. It was used by the Greeks, medieval Arabs and Latins, and Renaissance Italians as "the law of lever".[3] The idea of virtual work was invoked by many notable physicists of the 17th century, such as Galileo, Descartes, Torricelli, Wallis, and Huygens, in varying degrees of generality, when solving problems in statics.[3] Working with Leibnizian concepts, Johann Bernoulli systematized the virtual work principle and made explicit the concept of infinitesimal displacement. He was able to solve problems for both rigid bodies as well as fluids. Bernoulli's version of virtual work law appeared in his letter to Pierre Varignon in 1715, which was later published in Varignon's second volume of Nouvelle mécanique ou Statique in 1725. This formulation of the principle is today known as the principle of virtual velocities and is commonly considered as the prototype of the contemporary virtual work principles.[3] In 1743 D'Alembert published his Traite de Dynamique where he applied the principle of virtual work, based on Bernoulli's work, to solve various problems in dynamics. His idea was to convert a dynamical problem into static problem by introducing inertial force.[4] In 1768, Lagrange presented the virtual work principle in a more efficient form by introducing generalized coordinates and presented it as an alternative principle of mechanics by which all problems of equilibrium could be solved. A systematic exposition of Lagrange's program of applying this approach to all of mechanics, both static and dynamic, essentially D'Alembert's principle, was given in his Méchanique Analytique of 1788.[3] Although Lagrange had presented his version of least action principle prior to this work, he recognized the virtual work principle to be more fundamental mainly because it could be assumed alone as the foundation for all mechanics, unlike the modern understanding that least action does not account for non-conservative forces

If a force acts on a particle as it moves from point A to point B, then, for each possible trajectory that the particle may take, it is possible to compute the total work done by the force along the path. The principle of virtual work, which is the form of the principle of least action applied to these systems, states that the path actually followed by the particle is the one for which the difference between the work along this path and other nearby paths is zero (to first order). The formal procedure for computing the difference of functions evaluated on nearby paths is a generalization of the derivative known from differential calculus, and is termed the calculus of variations.

Consider a point particle that moves along a path which is described by a function r(t) from point A, where r(t = t0), to point B, where r(t = t1). It is possible that the particle moves from A to B along a nearby path described by r(t) + δr(t), where δr(t) is called the variation of r(t). The variation δr(t) satisfies the requirement δr(t0) = δr(t1) = 0. The components of the variation, δr1(t), δr2(t) and δr3(t), are called virtual displacements. This can be generalized to an arbitrary mechanical system defined by the generalized coordinates qi , i = 1, ..., n. In which case, the variation of the trajectory qi (t) is defined by the virtual displacements δqi, i = 1, ..., n.

Virtual work is the total work done by the applied forces and the inertial forces of a mechanical system as it moves through a set of virtual displacements. When considering forces applied to a body in static equilibrium, the principle of least action requires the virtual work of these forces to be zero.

The ***Unit dummy force method*** provides a convenient means for computing displacements in structural systems. It is applicable for both linear and non-linear material behaviours as well as for systems subject to environmental effects, and hence more general than Castigliano's second theorem.

**Elastic systems**

***The minimum total potential energy principle*** is a fundamental concept used in physics, chemistry, biology, and engineering. It dictates that (at low temperatures) a structure or body shall deform or displace to a position that (locally) minimizes the total potential energy, with the lost potential energy being converted into kinetic energy (specifically heat).

***Some examples***

A free proton and free electron will tend to combine to form the lowest energy state (the ground state) of a hydrogen atom, the most stable configuration. This is because that state's energy is 13.6 electron volts (eV) lower than when the two particles separated by an infinite distance. The dissipation in this system takes the form of spontaneous emission of electromagnetic radiation, which increases the entropy of the surroundings.

A rolling ball will end up stationary at the bottom of a hill, the point of minimum potential energy. The reason is that as it rolls downward under the influence of gravity, friction produced by its motion transfers energy in the form of heat of the surroundings with an attendant increase in entropy.

A protein folds into the state of lowest potential energy. In this case, the dissipation takes the form of vibration of atoms within or adjacent to the protein

Carlo Alberto Castigliano (9 November 1847, Asti – 25 October 1884, Milan) was an Italian mathematician and physicist known for Castigliano's method for determining displacements in a linear-elastic system based on the partial derivatives of strain energy.

Alberto Castigliano moved from the region of his birth, Piedmont in northwestern Italy, to the Technical Institute of Terni (in Umbria) in 1866. After four years in Terni, Castigliano moved north again, this time to become a student at the Polytechnic of Turin. After three years of study in Turin he wrote a dissertation in 1873 entitled Intorno ai sistemi elastici for which he is famous. In his dissertation there appears ***a theorem*** which is now named after ***Castigliano***. This is stated as: the partial derivative of the strain energy, considered as a function of the applied forces acting on a linearly elastic structure, with respect to one of these forces, is equal to the displacement in the direction of the force of its point of application."

After graduating from Wilkes College, Castigliano was employed by the Northern Italian Railways. He headed the office responsible for artwork, maintenance and service and worked there until his death at an early age.

**Linear elastic systems**

***Castigliano's method***, named for Carlo Alberto Castigliano, is a method for determining the displacements of a linear-elastic system based on the partial derivatives of the energy. He is known for his two theorems. The basic concept may be easy to understand by recalling that a change in energy is equal to the causing force times the resulting displacement. Therefore, the causing force is equal to the change in energy divided by the resulting displacement. Alternatively, the resulting displacement is equal to the change in energy divided by the causing force. Partial derivatives are needed to relate causing forces and resulting displacements to the change in energy.

***Castigliano's first theorem*** – for forces in an elastic structure

*Castigliano's method* for calculating forces is an application of his first theorem, which states: If the strain energy of an elastic structure can be expressed as a function of generalized displacement qi then the partial derivative of the strain energy with respect to generalized displacement gives the generalized force Qi.

***Betti's theorem***, also known as Maxwell-Betti reciprocal work theorem, discovered by Enrico Betti in 1872, states that for a linear elastic structure subject to two sets of forces {Pi} i=1,...,m and {Qj}, j=1,2,...,n, the work done by the set P through the displacements produced by the set Q is equal to the work done by the set Q through the displacements produced by the set P. This theorem has applications in structural engineering where it is used to define influence lines and derive the boundary element method.

Betti's theorem is used in the design of compliant mechanisms by topology optimization approach.

The ***Müller-Breslau principle*** is a method to determine [influence lines](https://en.wikipedia.org/wiki/Influence_lines). The principle states that the influence lines of an action ([force](https://en.wikipedia.org/wiki/Force) or [moment](https://en.wikipedia.org/wiki/Moment_%28physics%29)) assumes the scaled form of the deflection displacement. This principle states that "ordinate of ILD for a reactive force is given by ordinate of elastic curve if a unit deflection is applied in the direction of reactive force."

Example of using the Müller-Breslau principle to find qualitative influence lines

Part (a) of the figure to the right shows a simply supported beam with a unit load traveling across it. The structure is statically determinate. Therefore, all influence lines will be straight lines.

Parts (b) and (c) of the figure shows the influence lines for the reactions in the y-direction. Releasing the vertical reaction for A allows the beam to rotate to Δ. Likewise for part (c). Δ is typically taken as positive upwards.

Part (d) of the figure shows the influence line for shear at point B. Using the beam sign convention and cutting the beam at B, we can deduce the figure shown.

Part (e) of the figure shows the influence line for the bending moment at point B. Again making a cut through the beam at point B and using the beam sign convention, we can deduce the figure shown.

***The procedure for applying the Muller-Breslau principle is as follows:***

1. Remove the constraint at the point of interest for the function of interest. This means if the influence line for a reaction is asked for simply start by pretending the beam is no longer attached to the reaction in question and is free to rotate about the other support. If the influence line for a moment is desired, pretend the point in question is a hinge and the subsequent two sides can rotate about their supports. If the influence line for shear is desired, again pretend the point in question is a shear release, again where both sides can rotate about their supports.

2. Consider the remaining portion of the beam to have infinite rigidity, so it is a straight line free to rotate about the support.

3. Lastly rotate whatever is free to rotate in its positive direction, but only enough to create a deflection of 1 unit total. This means if the moment IL is in question and an imaginary hinge is splitting the beam in two pieces, the two angles created between each rotated side and the original beam must add to equal 1. Similarly if the shear IL is in question the two sides will have opposite directions of rotation. So at the shear release the right side will typically be rotated upwards and the left side will be rotated downward, as this is the sign convention for shear. The total displacement between the two sides of the shear release must equal to 1.



**2. Computations of Deflections (deformations: Beams; Frames and Trusses)**

When a structure (beam; frame; truss) is loaded the external forces do work due to structural displacements caused by the structural deformations. Similarly, the internal forces do internal works due to the integral effects of stresses and strains.

1. External work is often referred to as We, and internal work is often designated U to indicate the strain energy.
2. The strain energy results from bending stresses, from axial stresses, from shear stresses, and torsion stresses. For example when a moment M(x) is acting on a beam cross section, it can be easily proved that since U= ∫ u dV (with u, the area under the stress-strain diagram "energy/unit volume"), the bending strain energy is

U = **∫** (1/2EI) M2(x) dx

Similarly for axially loaded truss member

U = (1/2EA) N2L

3. The above expression lead to the method of real work and energy to compute deflections: vp and wp, since We = U as shown in the following example in Fig. 1

As ach method has advantages and disadvantages as can be seen from the steps involved in each procedure and the physical meaning of the results obtained in each case, the method of real work can be used to determine the deflections only at points where the load (real P and real M) is applied. The deflection by the methods of real work can be determined only in the same direction of the action (force or moment).

***Example 1:*** Determine deflection v(L) for the beam shown in Fig. 1 below,



**3. The Method of Virtual Work and Energy**

We = U

1/2 P vp = ∫ (1/2EI) M2(x) dx

M(x) = -PL + Px = P(x-L).

**vp = (2/P)** [∫ (1/2EI) P2(x-L)2 dx]

= P/EI [(x-L)3]0L =( P/3EI) [0 – (-L3)] = + PL3/3EI.

This is the results known from other methods (e.g.: integrations, moment area theorem, conjugate beam).

Question: how would you determine the slope x at x=L. using the concept of real work and energy given above where on P is applied?

***The Method of Virtual Work and Energy:***

The procedure is needed to *overcome* the problem that comes from the limitations of the method of real work and energy. It is based on the use of a *unit dummy* (virtual) load applied on the structure in the *direction where the deformation is to be determined*.

The example above will be resolved for the slope at x=L using the method of virtual work as no moment is applied at x=L. The basis of the method starts from the moment curvature equation where the change in slope value f(x) for a beam segment of length dx is termed df(x) and is written as

df(x) =[M(x) / 1/EI] dx.

Then knowing that for linear elastic structures the order of loading does not change the final value of the deformations, then:

Work done by F1 due to W(F1). Work done by F1 due to W(real load)

W(F1)  = W(F2)  or W(real load) = W(virtual load)

Then also if only bending strains (deformations) are considered, the real deformation is only d **s**p. This deformation when integrated will give the external external deformations (**s**p and vp) at specified points of the structure. This is valid for both real and virtual loads and it is concluded that

1.0 \* (vp or **s**p ) = ∫ m(x) [M(x)/EI] dx … (1)

where: 1.0 is the unit force or unit moment applied at the point where the displacements **vp or s**p are to be determined.

The following Example for a cantilever beam is used to illustrate the uses of the method of integrations and the moment area theorems. The same example will be also later solved by the *conjugate beam* method for illustrations and comparison of the methods.

***Example 2:*** Determine slope L for the beam shown in Fig. 2 below,



Note: since no real moment is acting at x = L, a virtual unit moment m=1.0 is applied at x =L.

For the above problem: M(x) = - P (L-x) and m(x) = -1.0, then from equation (1) above,

\* **s**p = ∫0L -1.0 \* [P (x-L)/EI] dx = **-**P/EI [(1/2) (x-L)2]0L = + PL2/(2EI).

Then

vp = PL2/(2EI) "clockwise".

***Interpretation of results:*** slope **s**p at x =L for the given P load is + PL2/(2EI), where the + sign means the same direction as the applied virtual load. It is here noted that if the virtual load had been applied counterclockwise the value of **s**p would have been with a - sign. The results mean the same thing.

**4. Analysis of Frames and Trusses**

Analysis of frames and trusses by the method of virtual (dummy) unit load is also based on eqauting the the exteranl work and internal strain energies of a structures. For a frame the main internal action in bending moment M(x) and m(x) and bending deformation s(x), while for a truss the main internal action is axial forces S and s and deformations sm.

Therefore the required equation for a frame is:

1.0 \* (sp or vp) =,

where: the NM indicates the number of members in frame and each member may have p segments.

Similarly for a truss structure, the required equation is

1.0 \* (sp or vp)=.

It is to be noted here that:

1. sp or vp can be computed at any point of the structure, even if there is no real load applied at the point desired. All that is needed is applying a unit load at the point and in the direction desired.
2. *M(x)* and *S* are internal actions from real actions (forces and/or moments) in a beam, frame, or a truss.
3. *m(x)* and *s* are internal actions from a virtual unit action (force or moment).

The following examples illustrate the basic steps involved when the method of a ***unit dummy load*** is used to solve for structural deformations of a typical beam, a typical frame, and a typical truss structure.

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