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**Курс лекций**

**по дисциплине «Гамильтонова механика и качественные особенности движения тела (на английском)»**

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The Faculty of Mathematics and Information Technologies

The Chair of Algebra, Mathematical Logic and Geometry named after Professor Mustafin T.G.

**Yessenbayeva Gulsim Akhmadievna**

**The course of lectures**

**by the discipline «** **Hamiltonian mechanics and qualitative features of body motion (in English)»**

Educational program: «7M05402 Mechanics»

Karaganda 2022

**Lecture 1**

## Lecture topic: [Representation of the Angular Velocity Vector](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_2.6). Euler Angles

**The plan**

## [1. Representation of the Angular Velocity Vector](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_2.6)

## [2. Euler Angles](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_2.7)

## [3. Vector Angular Momentum](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_2.8)

## [4. Motion of a Free Rigid Body](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_2.9)

## [1. Representation of the Angular Velocity Vector](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_2.6)

We can specify the orientation of a body by specifying the rotation that takes the body to this orientation from some reference orientation. As the body moves, the rotation that does this changes. The angular velocity vector can be written in terms of this changing rotation along a path.



Let q be the coordinate path that we will use to describe the motion of the body. Let M(q(t)) be the rotation that takes the body from the reference orientation to the orientation specified by q(t) (see figure [2.1](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-26.html#FIGURE_2.1)). Let https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-3.gifhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-21.gif(t) be the vector to some constituent particle with the body in the orientation specified by q(t), and let https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-3.gifhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-21.gif' be the vector to the same constituent with the body in the reference orientation. Then

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-34.gif

The constituent vectors https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-3.gifhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-21.gif/ do not depend on the configuration, because they are the vectors to the positions of the constituents with the body in a fixed reference orientation.

We have already found an expression for the kinetic energy in terms of the angular velocity vector and the inertia tensor. Here we do this in a different way. To compute the kinetic energy we accumulate the contributions from all of the mass elements. The positions of the constituent particles, at a given time t, are

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where M = M o q. The velocity is the time derivative

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-36.gif

Using equation ([2.32](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-26.html#EQUATION_2.32)), we can write

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Recall that the velocity results from a rotation, and that the velocities are (see equation [2.11](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-22.html#EQUATION_2.11))

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-38.gif

Thus we can identify the operator https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-8.gif(t) × with DM(t) (M(t))-1. To form the kinetic energy we need to extract https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-8.gif(t) from this.

If a vector https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-15.gifis represented by the component matrix **u** with components x, y, and z, the function A that produces the matrix representation of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-15.gif× from the component matrix **u** is

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The inverse of this function can be applied to any skew-symmetric matrix, and so we can use A-1 to extract the components https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-23.gifof the angular velocity vector from the matrix representation of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-8.gif× in terms of M:

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-40.gif

where **M** and D**M** are the matrix representations of the functions M and DM, and where we have used the fact that for a matrix representation of a rotation the transpose gives the inverse.

The components https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-23.gif' of the angular velocity vector on the principal axes are https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-23.gif' = **M**T https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-23.gif, so

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The relationship of the angular velocity vector to the path is a kinematic relationship; it is valid for any path. Thus we can abstract it to obtain the components of the angular velocity at a moment given the configuration and velocity at that moment.

#### [Implementation of angular velocity functions](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html" \l "%_toc_%_sec_Temp_190)

The following procedure gives the components of the angular velocity as a function of time along the path:

The procedure omega-cross produces the matrix representation of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-8.gif×. The procedure antisymmetric->column-matrix, which corresponds to the function A-1, is used to extract the components of the angular velocity vector from the skew-symmetric https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-8.gif× matrix.

The body components of the angular velocity vector as a function of time along the path are

We can get the procedures of local state that give the angular velocity components by abstracting these procedures along arbitrary paths that have given coordinates and velocities. The abstraction of a procedure of a path to a procedure of state is accomplished by Gamma-bar (see section [1.9](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-16.html#%_sec_1.9)):

These procedures give the angular velocities as a function of state. We will see them in action after we get some M-of-q's with which to work.

## [2. Euler Angles](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_2.7)

To go further we must finally specify a set of generalized coordinates. We first do this using the traditional Euler angles. Later, we find other ways of describing the orientation of a rigid body.

We are using an intermediate representation of the orientation in terms of the function M of the generalized coordinates that gives the rotation that takes the body from some reference orientation and rotates it to the orientation specified by the generalized coordinates. Here we take the reference orientation so that principal-axis unit vectors hata, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-13.gif, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-14.gifare coincident with the basis vectors https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-9.gifi, labeled here by https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-6.gif, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-16.gif, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-7.gif.

We define the Euler angles in terms of simple rotations about the coordinate axes. Let Rx(https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-56.gif) be a right-handed rotation about the https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-6.gifaxis by the angle https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-56.gif, and let Rz(https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-56.gif) be a right-handed rotation about the https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-7.gifaxis by the angle https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-56.gif. The function M for Euler angles is written as a composition of three of these simple coordinate axis rotations:

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for the Euler angles https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gif, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-56.gif.

The Euler angles can specify any orientation of the body, but the orientation does not always correspond to a unique set of Euler angles. In particular, if https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif= 0 then the orientation is dependent only on the sum https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gif+ https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-56.gif, so the orientation does not uniquely determine either https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gifor https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-56.gif.

**Exercise 2.8.**  **Euler angles**

It is not immediately obvious that all orientations can be represented in terms of the Euler angles. To show that the Euler angles are adequate to represent all orientations, solve for the Euler angles that give an arbitrary rotation R. Keep in mind that some orientations do not correspond to a unique representation in terms of Euler angles.

Though the Euler angles allow us to specify all orientations and thus can be used as generalized coordinates, the definition of Euler angles is pretty arbitrary. In fact no reasoning has led us to them. This is reflected in our presentation of them by just saying ``here they are.'' Euler angles are well suited for some problems, but cumbersome for others.

There are other ways of defining similar sets of angles. For instance, we could also take our generalized coordinates to satisfy

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Such alternatives to the Euler angles come in handy from time to time.

Each of the fundamental rotations can be represented as a matrix. The rotation matrix representing a right-handed rotation about the https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-7.gifaxis by the angle https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-56.gifis

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-44.gif

and a right-handed rotation about the x axis by the angle https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-56.gifis represented by the matrix

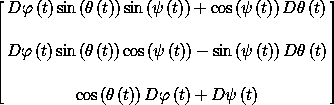
https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-45.gif

The matrix that represents the rotation that carries the body from its reference orientation to the actual orientation is

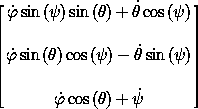
https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-46.gif

The rotation matrices and their product can be constructed by simple programs:

Now that we have a procedure that implements a sample M, we can find the components of the angular velocity vector and the body components of the angular velocity vector using the procedures M-of-q->omega-of-t and M-of-q->omega-body-of-t from section [2.6](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-26.html#%_sec_2.6). For example,



To construct the kinetic energy we need the procedure of state that gives the body components of the angular velocity vector:



## [3. Vector Angular Momentum](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_2.8)

The vector angular momentum of a particle is the cross product of the position and the linear momentum. For a rigid body the vector angular momentum is the sum of the vector angular momentum of each of the constituents. Here we find an expression for the vector angular momentum of a rigid body in terms of the inertia tensor and the angular velocity vector.

The vector angular momentum of a rigid body is

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where https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-26.gifhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-21.gif, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-1.gifhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-21.gif, and mhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-21.gif are the positions, velocities, and masses of the constituent particles. It turns out that the vector angular momentum decomposes into the sum of the angular momentum of the center of mass and the rotational angular momentum about the center of mass, just as the kinetic energy separates into the kinetic energy of the center of mass and the kinetic energy of rotation. As in the kinetic energy demonstration, decompose the position into the vector to the center of mass https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-2.gifand the vectors from the center of mass to the constituent mass elements https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-3.gifhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-21.gif:

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-50.gif

with velocities

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-51.gif

Substituting, the angular momentum is

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Multiplying out the product, and using the fact that https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-2.gifis the center of mass and M = sumhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-21.gif mhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-21.gif is the total mass of the body, the angular momentum is

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-53.gif

The angular momentum of the center of mass is

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and the rotational angular momentum is

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We can also reexpress the rotational angular momentum in terms of the angular velocity vector and the inertia tensor, as we did for the kinetic energy. Using https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-5.gifhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-21.gif= https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-8.gif× https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-3.gifhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-21.gif, we get the rotational angular momentum

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-56.gif

In terms of components with respect to the basis {https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-9.gif0, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-9.gif1, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-9.gif2}, this is

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-57.gif

where Ijk are the components of the inertia tensor ([2.14](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-23.html#EQUATION_2.14)). The angular momentum and the kinetic energy are expressed in terms of the same inertia tensor.

With respect to the principal-axis basis, the angular momentum components have a particularly simple form:



**Exercise 2.9.**

Verify that expression ([2.52](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-28.html#EQUATION_2.52)) for the components of the rotational angular momentum ([2.51](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-28.html#EQUATION_2.51)) in terms of the inertia tensor is correct.

We can define procedures to calculate the components of the angular momentum on the principal axes:

We then transform the components of the angular momentum on the principal axes to the components on the fixed basis https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-9.gifi:

These procedures are local state functions, like Lagrangians.

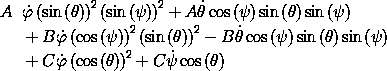
## [4. Motion of a Free Rigid Body](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_2.9)

The kinetic energy, expressed in terms of a suitable set of generalized coordinates, is a Lagrangian for a free rigid body. In section [2.1](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-21.html#%_sec_2.1) we found that the kinetic energy of a rigid body can be written as the sum of the rotational kinetic energy and the translational kinetic energy. If we choose one set of coordinates to specify the position and another set to specify the orientation, the Lagrangian becomes a sum of a translational Lagrangian and a rotational Lagrangian. The Lagrange equations for translational motion are not coupled to the Lagrange equations for the rotational motion. For a free rigid body the translational motion is just that of a free particle: uniform motion. Here we concentrate on the rotational motion of the free rigid body. We can adopt the Euler angles as the coordinates that specify the orientation; the rotational kinetic energy was expressed in terms of Euler angles in the previous section.

#### [Conserved quantities](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html" \l "%_toc_%_sec_Temp_193)

The Lagrangian for a free rigid body has no explicit time dependence, so we can deduce that the energy, which is just the kinetic energy, is conserved by the motion.

The Lagrangian does not depend on the Euler angle https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gif, so we can deduce that the momentum conjugate to this coordinate is conserved. An explicit expression for the momentum conjugate to https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gifis



We know that this complicated quantity is conserved by the motion of the rigid body because of the symmetries of the Lagrangian.

If there are no external torques, then we expect that the vector angular momentum will be conserved. We can verify this using the Lagrangian formulation of the problem. First, we note that Lz is the same as phttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gif. We can check this by direct calculation:

We know that phttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gif is conserved because the Lagrangian for the free rigid body did not mention https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gif, so now we know that Lz is conserved. Since the orientation of the coordinate axes is arbitrary, we know that if any rectangular component is conserved then all of them are. So the vector angular momentum is conserved for the free rigid body.

We could have seen this with the help of Noether's theorem (see section [1.8.4](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-15.html#%_sec_1.8.4)). There is a continuous family of rotations that can transform any orientation into any other orientation. The orientation of the coordinate axes we used to define the Euler angles is arbitrary, and the kinetic energy (the Lagrangian) is the same for any choice of coordinate system. Thus the situation meets the requirements of Noether's theorem, which tells us that there is a conserved quantity. In particular, the family of rotations around each coordinate axis gives us conservation of the angular momentum component on that axis. We construct the vector angular momentum by combining these contributions.

**Exercise 2.10.**  **Vector angular momentum**

Fill in the details of the argument that Noether's theorem implies that vector angular momentum is conserved by the motion of the free rigid body.

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### [*Computing the Motion of Free Rigid Bodies*](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_2.9.1)

Lagrange's equations for the motion of a free rigid body in terms of Euler angles are quite disgusting, so we will not show them here. However, we will use the Lagrange equations to explore the motion of the free rigid body.

Before doing this it is worth noting that the equations of motion in Euler angles are singular for some configurations, because for these configurations the Euler angles are not uniquely defined. If we set https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif= 0 then an orientation does not correspond to a unique value of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gifand https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-56.gif; only their sum determines the orientation.

The singularity arises in the explicit Lagrange equations when we attempt to solve for the second derivative of the generalized coordinates in terms of the generalized coordinates and the generalized velocities (see section [1.7](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-14.html#%_sec_1.7)). The isolation of the second derivative requires multiplying by the inverse of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif2 https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif2 L. The determinant of this quantity becomes zero when the Euler angle https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gifis zero:

So when https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gifis zero, we cannot solve for the second derivatives. When https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gifis small, the Euler angles can move very rapidly, and thus may be difficult to compute reliably. Of course, the motion of the rigid body is perfectly well behaved for any orientation. This is a problem of the representation of that motion in Euler angles; it is a ``coordinate singularity.''

One solution to this problem is to use another set of Euler-like coordinates for which Lagrange's equations have singularities for different orientations, such as those defined in equation ([2.40](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-27.html#EQUATION_2.40)). So if as the calculation proceeds the trajectory comes close to a singularity in one set of coordinates, we can switch coordinate systems and use another set for a while until the trajectory encounters another singularity. This solves the problem, but it is cumbersome. For the moment we will ignore this problem and compute some trajectories, being careful to limit our attention to trajectories that avoid the singularities.

We will compute some trajectories by numerical integration and check our integration process by seeing how well energy and angular momentum are conserved. Then, we will investigate the evolution of the components of angular momentum on the principal axis basis. We will discover that we can learn quite a bit about the qualitative behavior of rigid bodies by combining the information we get from the energy and angular momentum.

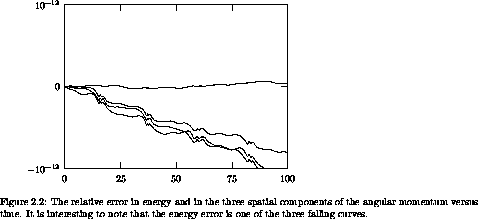
To develop a trajectory from initial conditions we integrate the Lagrange equations, as we did in chapter [1](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-7.html#%_chap_1). The system derivative is obtained from the Lagrangian:

The following program monitors the errors in the energy and the components of the angular momentum:

We make a plot window to display the errors:

The default integration method used by the system is Bulirsch-Stoer (bulirschstoer), but here we set the integration method to be quality-controlled Runge-Kutta (qcrk4), because the error plot is more interesting:

The plot that is developed of the relative errors in the components of the angular momenta and the energy (see figure [2.2](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-29.html#FIGURE_2.2)) shows that we have been successful in controlling the error in the conserved quantities. This should give us some confidence in the trajectory that is evolved.



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**Lecture 2**

### Lecture topic: [Qualitative Features of Free Rigid Body Motion](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_2.9.2)

**The plan**

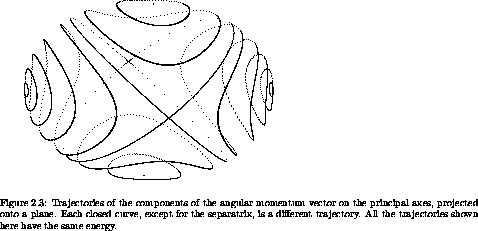
### [1. Qualitative Features of Free Rigid Body Motion](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_2.9.2)

## [2. Axisymmetric Tops](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_2.10)

### [1. Qualitative Features of Free Rigid Body Motion](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_2.9.2)

The evolution of the components of the angular momentum on the principal axes has a remarkable property. For almost every initial condition the body components of the angular momentum periodically trace a simple closed curve.

We can see this by investigating a number of trajectories and plotting the components of angular momentum of the body on the principal axes (see figure [2.3](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-29.html#FIGURE_2.3)). To make this figure a number of trajectories of equal energy were computed. The three-dimensional space of body components is projected onto a two-dimensional plane for display. Points on the back of this projection of the ellipsoid of constant energy are plotted with lower density than points on the front of the ellipsoid. For most initial conditions we find a one-dimensional simple closed curve. Some trajectories on the front side appear to cross trajectories on the back side, but this is an artifact of projection. There is also a family of trajectories that appear to intersect in two points, one on the front side and one on the back side. The curve that is the union of these trajectories is called a separatrix; it separates different types of motion.



What is going on? The state space for a free rigid body is six-dimensional: the three Euler angles and their time derivatives. We know four constants of the motion -- the three spatial components of the angular momentum, Lx, Ly, and Lz, and the energy, E. Thus, the motion is restricted to a two-dimensional region of the state space.[9](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-29.html" \l "footnote_Temp_195) Our experiment shows that the components of the angular momentum trace one-dimensional closed curves in the angular-momentum subspace, so there is something more going on here.

The total angular momentum is conserved if all of the components are, so we also have the constant

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The spatial components of the angular momentum do not change, but of course the projections of the angular momentum onto the principal axes do change because the axes move as the body moves. However, the magnitude of the angular momentum vector is the same whether it is computed from components on the fixed basis or components on the principal axis basis. So, the combination

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is conserved.

Using the expressions ([2.53](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-28.html#EQUATION_2.53)-[2.55](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-28.html#EQUATION_2.55)) for the angular momentum in terms of the components of the angular velocity vector on the principal axes, the kinetic energy ([2.30](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-25.html#EQUATION_2.30)) can be rewritten in terms of the angular momentum components on the principal axes:

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The two conserved quantities ([2.57](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-29.html#EQUATION_2.57) and [2.58](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-29.html#EQUATION_2.58)) provide constraints on how the components of the angular momentum vector on the principal axes can change. We recognize the angular momentum integral ([2.57](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-29.html#EQUATION_2.57)) as the equation of a sphere, and the kinetic energy integral ([2.58](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-29.html#EQUATION_2.58)) as the equation for a triaxial ellipsoid. Both integrals are conserved so the components of the angular momentum are constrained to move on the intersection of these two surfaces, the energy ellipsoid and the angular momentum sphere. The intersection of an ellipsoid and a sphere with the same center is generically two closed curves, so an orbit is confined to one of these curves. This sheds light on the puzzle posed at the beginning of this section.

Because of our ordering A < B < C, the longest axis of this triaxial ellipsoid coincides with the https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-14.gifdirection if all the angular momentum is along the axis of largest principal moment of inertia, and the shortest axis of the energy ellipsoid coincides with the hata axis if all the angular momentum is along the smallest moment of inertia. Without actually solving the Lagrange equations, we have found strong constraints on the evolution of the components of the angular momentum on the principal axes.

To determine how the system evolves along these intersection curves we have to use the equations of motion. We observe that the evolution of the components of the angular momentum on the principal axes depends only on the components of the angular momentum on the principal axes, even though the values of these components are not enough to completely specify the dynamical state. Apparently the dynamics of these components is self-contained, and we will see that it can be described in terms of a set of differential equations whose only dynamical variables are the components of the angular momentum on the principal axes (see section [2.12](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-32.html#%_sec_2.12)).

We note that there are two axes for which the intersection curves shrink to a point if we hold the energy constant and vary the magnitude of the angular momentum. If the angular momentum starts at these points, the integrals constrain the angular momentum to stay there. These points are equilibrium points for the body components of the angular momentum. However, they are not equilibrium points for the system as a whole. At these points the body is still rotating even though the body components of the angular momentum are not changing. This kind of equilibrium is called a relative equilibrium. We can also see that if the angular momentum is initially slightly displaced from one of these relative equilibria, then the angular momentum is constrained to stay near it on one of the intersection curves. The angular momentum vector is fixed in space, so the principal axis of the equilibrium point of the body rotates stably about the angular momentum vector.

At the principal axis with intermediate moment of inertia, the https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-13.gifaxis, the intersection curves cross. As we observed, the dynamics of the components of the angular momentum on the principal axes forms a self-contained dynamical system. Trajectories of a dynamical system cannot cross,[10](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-29.html" \l "footnote_Temp_196) so the most that can happen is that if the equations of motion carry the system along the intersection curve then the system can approach the crossing point only asymptotically. So without solving any equations we can deduce that the point of crossing is another relative equilibrium. If the angular momentum is initially aligned with the intermediate axis, then it stays aligned. If the system is slightly displaced from the intermediate axis, then the evolution along the intersection curve will take the system far from the relative equilibrium. So rotation about the axis of intermediate moment of inertia is unstable -- initial displacements of the angular momentum, however small initially, become large. Again, the angular momentum vector is fixed in space, but now the principal axis with the intermediate principal moment does not stay close to the angular momentum, so the body executes a complicated tumbling motion.

This gives some insight into the mystery of the thrown book mentioned at the beginning of the chapter. If one throws a book so that it is initially rotating about either the axis with the largest moment of inertia or the axis with the smallest moment of inertia (the smallest and largest physical axes, respectively), the book rotates regularly about that axis. However, if the book is thrown so that it is initially rotating about the axis of intermediate moment of inertia (the intermediate physical axis), then it tumbles, however carefully it is thrown. You can try it with this book (but put a rubber band or string around it first).

Before moving on, we can make some further physical deductions. Suppose a freely rotating body is subject to some sort of internal friction that dissipates energy but conserves the angular momentum. For example, real bodies flex as they spin. If the spin axis moves with respect to the body then the flexing changes with time, and this changing distortion converts kinetic energy of rotation into heat. Internal processes do not change the total angular momentum of the system. If we hold the magnitude of the angular momentum fixed but gradually decrease the energy, then the curve of intersection on which the system moves gradually deforms. For a given angular momentum there is a lower limit on the energy: the energy cannot be so low that there are no intersections. For this lowest energy the intersection of the angular momentum sphere and the energy ellipsoid is a pair of points on the axis of maximum moment of inertia. With energy dissipation, a freely rotating physical body eventually ends up with the lowest energy consistent with the given angular momentum, which is rotation about the principal axis with the largest moment of inertia (typically the shortest physical axis).

Thus, we expect that given enough time all freely rotating physical bodies will end up rotating about the axis of largest moment of inertia. You can demonstrate this to your satisfaction by twirling a small bottle containing some viscous fluid, such as correction fluid. What you will find is that, whatever spin you try to put on the bottle, it will reorient itself so that the axis of the largest moment of inertia is aligned with the spin axis. Remarkably, this is very nearly true of almost every body in the solar system for which there is enough information to decide. The deviations from principal axis rotation for the Earth are tiny: the angle between the angular momentum vector and the https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-14.gifaxis for the Earth is less than one arc-second.[11](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-29.html" \l "footnote_Temp_197) In fact, the evidence is that all of the planets, the Moon and all the other natural satellites, and almost all of the asteroids rotate very nearly about the largest moment of inertia. We have deduced that this is to be expected using an elementary argument. There are exceptions. Comets typically do not rotate about the largest moment. As they are heated by the sun, material spews out from localized jets, and the back reaction from these jets changes the rotation state. Among the natural satellites, the only known exception is Saturn's satellite Hyperion, which is tumbling chaotically. Hyperion is especially out of round and subject to strong gravitational torques from Saturn.

## [2. Axisymmetric Tops](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_2.10)

We have all played with a top at one time or another. For the purposes of analysis we will consider an idealized top that does not wander around. Thus, an ideal top is a rotating rigid body, one point of which is fixed in space. Furthermore, the center of mass of the top is not at the fixed point, which is the center of rotation, and there is a uniform gravitational acceleration.

For our top we can take the Lagrangian to be the difference of the kinetic energy and the potential energy. We already know how to write the kinetic energy -- what is new here is that we must express the potential energy in terms of the configuration. In the case of a body in a uniform gravitational field this is easy. The potential energy is the sum of ``mgh'' for all the constituent particles:

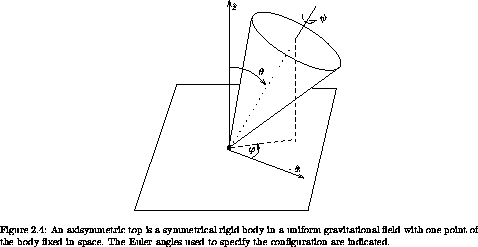
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where g is the gravitational acceleration, hhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-21.gif = https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-26.gifhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-21.gif· https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-7.gif, and the unit vector https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-7.gifindicates which way is up. Rewriting the vector to the constituents in terms of the vector https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-2.gifto the center of mass, the potential energy is



where the last sum is zero because the center of mass is the origin of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-3.gifhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-21.gif. So the potential energy of a body in a gravitational field with uniform acceleration is very simple: it is just Mgh, where M is the total mass and h = https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-2.gif· https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-7.gifis the height of the center of mass.

Here we consider an axisymmetric top (see figure [2.4](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-30.html#FIGURE_2.4)). Such a top has an axis of symmetry of the mass distribution, so the center of mass is on the symmetry axis and the fixed point is also on the axis of symmetry.



In order to write the Lagrangian we need to choose a set of generalized coordinates. If we choose them well we can take advantage of the symmetries of the problem. If the Lagrangian does not depend on a particular coordinate, the conjugate momentum is conserved, and the complexity of the system is reduced.

The axisymmetric top has two apparent symmetries. The fact that the mass distribution is axisymmetric implies that neither the kinetic nor potential energy is sensitive to the orientation of the top about that symmetry axis. Additionally, the kinetic and potential energy are insensitive to a rotation of the physical system about the vertical axis, because the gravitational field is uniform.

We can take advantage of these symmetries by choosing appropriate coordinates, and we already have a coordinate system that does the job -- the Euler angles.[12](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-30.html" \l "footnote_Temp_198) We choose the reference orientation so that the symmetry axis is vertical. The first Euler angle, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-56.gif, expresses a rotation about the symmetry axis. The next Euler angle, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif, is the tilt of the symmetry axis of the top from the vertical. The third Euler angle, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gif, expresses a rotation of the top about the z axis. The symmetries of the problem imply that the first and third Euler angles do not appear in the Lagrangian. As a consequence the momenta conjugate to these angles are conserved quantities. Let's work out the details.

First, we develop the Lagrangian explicitly. The general form of the kinetic energy about a point is given by equation [2.30](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-25.html#EQUATION_2.30). The top is constrained so that it pivots about a fixed point that is not at the center of mass. So the moments of inertia that enter the kinetic energy are the moments of inertia of the top with respect to the pivot point, not with respect to the center of mass. If we know the moments of inertia about the center of mass we can write the moments of inertia about the pivot in terms of them (see exercise [2.2](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-23.html#%_thm_2.2)). So let's assume the principal moments of inertia of the top about the pivot are A, B, and C, and A = B because of the symmetry.[13](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-30.html" \l "footnote_Temp_199) We can use the computer to help us figure out the Lagrangian for this special case:

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We can rearrange this a bit to get

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In terms of Euler angles, the potential energy is

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where R is the distance of the center of mass from the pivot. The Lagrangian is L = T - V. We see that the Lagrangian is indeed independent of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-56.gifand https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gif, as expected.

There is no particular reason to look at the Lagrange equations. We can assign that job to the computer when needed. However, we have already seen that it may be useful to examine the conserved quantities associated with the symmetries.

The energy is conserved, because the Lagrangian has no explicit time dependence. Also, the energy is the sum of the kinetic and potential energy E = T + V, because the kinetic energy is a homogeneous quadratic form in the generalized velocities. The energy is

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Two of the generalized coordinates do not appear in the Lagrangian, so there are two conserved momenta. The momentum conjugate to https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gifis

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The momentum conjugate to https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-56.gifis

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-74.gif

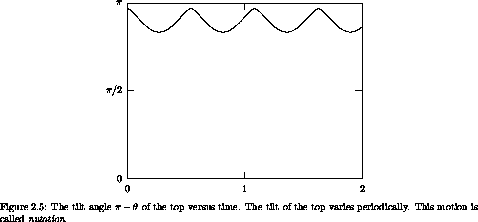
The state of the system at a moment is specified by the tuple ( t; https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gif, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-56.gif; https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-20.gif, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-34.gif, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-17.gif). The two coordinates https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gifand https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-56.gifthat do not appear in the Lagrangian do not appear in the Lagrange equations or the conserved momenta. So the evolution of the remaining four state variables, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-20.gif, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-34.gif, and https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-17.gif, depends only on those remaining state variables. This subsystem for the top has a four-dimensional state space. The variables that did not appear in the Lagrangian can be determined by integrating the derivatives of these variables, which are determined separately by solving the independent subsystem.

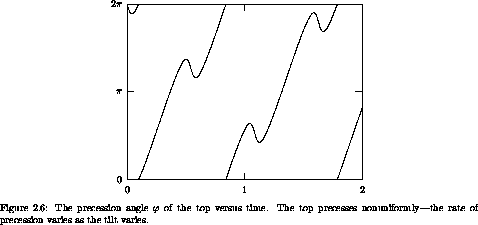
The evolution of the top is described by a four-dimensional subsystem and two auxiliary quadratures.[14](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-30.html" \l "footnote_Temp_200) This subdivision is a consequence of choosing generalized coordinates that incorporate the symmetries. However, the choice of generalized coordinates that incorporate the symmetries also gives conserved momenta. We can make use of these momenta to simplify the formulation of the problem further. Each integral can be used to locally eliminate one dimension of the subsystem. In this case the subsystem has four dimensions and there are three integrals, so the system can be completely reduced to quadratures. For the top, this can be done analytically, but we think it is a waste of time to do so. Rather, we are interested in extracting interesting features of the motion. We concentrate on the energy integral and use the two conserved momenta to eliminate https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-34.gifand https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-17.gif. After a bit of algebra we find:

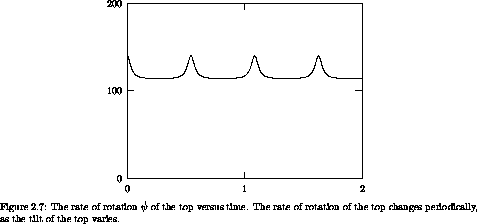
https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-75.gif

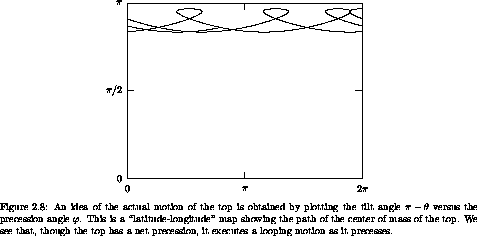
Along a path https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif, where Dhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif(t) is substituted for https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-20.gif, this is an ordinary differential equation for https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif. This differential equation involves various constants, some of which are set by the initial conditions of the other state variables. The solution of the differential equation for https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gifinvolves no more than ordinary integrals. So the top is essentially solved. We could continue this argument to obtain the qualitative behavior of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif: Using the energy ([2.66](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-30.html#EQUATION_2.66)), we can plot the trajectories in the plane of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-20.gifversus https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gifand see that the motion of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gifis simply periodic. However, we will defer continuing along this path until chapter [3](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-36.html#%_chap_3), when we have developed more tools for analysis.

Let's get real. Let's make a top out of an aluminum disk with a steel rod through the center to make the pivot. Measuring the top very carefully, we find that the moment of inertia of the top about the symmetry axis is about 6.60 × 10-5 kg m2, and the moment of inertia about the pivot point is about 3.28 × 10-4 kg m2. The combination gMR is about 0.0456 kg m2 s-2. We spin the top up with an initial angular velocity of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-17.gif= 140 rad s-1 (about 1337 rpm). The top initially has https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-20.gif= https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gif= https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-56.gif= 0 and is initially tilted with https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif= 0.1 rad. We then kick it so that https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-34.gif= - 15 rad s-1. Figures [2.5](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-30.html#FIGURE_2.5)-[2.8](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-30.html#FIGURE_2.8) display aspects of the evolution of the top for 2 seconds. The tilt of the top (measured by https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif) varies in a periodic manner. The orientation about the vertical is measured by https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gif: we see that the top also precesses, and the rate of precession varies with https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif. We also see that as the top bobs up and down the rate of rotation of the top oscillates -- the top spins faster when it is more vertical. The plot of tilt versus precession angle shows that in this case the top executes a looping motion. If we do not kick it but just let it drop, then the loop disappears, leaving just a cusp. If we kick it in the other direction, then there is no cusp nor any looping motion.









**Exercise 2.11.**  **Kinetic energy of the top**

The rotational kinetic energy of the top can be written in terms of the principal moments of inertia with respect to the pivot point and the angular velocity vector of rotation with respect to the pivot point. Show that this formulation of the kinetic energy yields the same value that one would obtain by computing the sum of the rotational kinetic energy about its center of mass and the kinetic energy of the motion of the center of mass.

**Exercise 2.12.**  **Nutation of the top**

**a**.  Carry out the algebra to obtain the energy ([2.66](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-30.html#EQUATION_2.66)) in terms of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gifand https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-20.gif.

**b**.  Numerically integrate the Lagrange equations for the top to obtain figure [2.5](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-30.html#FIGURE_2.5), https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gifversus time.

**c**.  Note that the energy is a differential equation for https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-20.gifin terms of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif, with conserved quantities phttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gif, phttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-56.gif, and E determined by initial conditions. Can we use this differential equation to obtain https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gifas a function of time? Explain.

**Exercise 2.13.**  **Precession of the top**

Consider a top that is rotating so that https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gifis constant.

**a**.  Using the angular momentum integrals, compute the rate of precession https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-34.gifas a function of the conserved angular momenta and the equilibrium value of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif.

**b**.  For https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gifto be at an equilibrium the acceleration D2 https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gifmust be zero. Use the Lagrange equation for https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gifto find the rate of precession https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-34.gifat the equilibrium in terms of the equilibrium https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gifand https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-17.gif.

**c**.  Find an approximate expression for the precession rate in the limit that https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap2-Z-G-D-17.gifis large.

**d**.  The Newtonian rule is that the rate of change of the angular momentum is the torque. Assume the top is spinning so fast that the angular momentum is nearly the same as the angular momentum of rotation about the symmetry axis. By equating the rate of change of this vector angular momentum to the gravitational torque on the center of mass develop an approximate formula for the precession rate.

**e**.  Numerically integrate the top to check your deductions.

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**Lecture 3**

## Lecture topic: [[Hamiltonian Mechanics](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html" \l "%_toc_%_chap_1)](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_chap_3)[.](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html" \l "%_toc_%_chap_1) [[Hamilton's Equations](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html" \l "%_toc_%_chap_1)](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_3.1)

**The plan**

1. [Hamiltonian Mechanics](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_chap_3)

## [2. Hamilton's Equations](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_3.1)

### [3. The Legendre Transformation](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_3.1.1)

**1.** [**Hamiltonian Mechanics**](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_chap_3)

|  |
| --- |
|  |

The formulation of mechanics with generalized coordinates and momenta as dynamical state variables is called the Hamiltonian formulation. The Hamiltonian formulation of mechanics is equivalent to the Lagrangian formulation; however, each presents a useful point of view. The Lagrangian formulation is especially useful in the initial formulation of a system. The Hamiltonian formulation is especially useful in understanding the evolution, especially when there are symmetries and conserved quantities.

For each continuous symmetry of a mechanical system there is a conserved quantity. If the generalized coordinates can be chosen to reflect a symmetry, then, by the Lagrange equations, the conjugate momentum is conserved. We have seen that such conserved quantities allow us to deduce important properties of the motion. For instance, consideration of the energy and angular momentum allowed us to deduce that rotation of a free rigid body about the axis of intermediate moment of inertia is unstable, and that rotation about the other principal axes is stable. For the axisymmetric top, we used two conserved momenta to reexpress the equations governing the evolution of the tilt angle so that they involve only the tilt angle and its derivative. The evolution of the tilt angle can be determined independently and has simply periodic solutions. Consideration of the conserved momenta has provided key insight. The Hamiltonian formulation is motivated by the desire to focus attention on the momenta.

## [2. Hamilton's Equations](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_3.1)

The equations of motion when recast in terms of coordinates and momenta are called Hamilton's canonical equations.

Lagrange's equations give us the time derivative of the momentum p on a path q:

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-1.gif

where

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-2.gif

To eliminate Dq we need to solve equation ([3.2](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.2)) for Dq in terms of p.

Let https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gifbe the function that gives the velocities in terms of the time, coordinates, and momenta. Defining https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gifis a problem of functional inverses. To prevent confusion we use names for the variables that have no mnemonic significance. Let

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-3.gif

then https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gif satisfies

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-4.gif

So https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gifand https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif2 L are inverses on the third argument position:

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-5.gif

The Lagrange equation ([3.1](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.1)) can be rewritten in terms of p using https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gif:

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-6.gif

We can also use https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gifto rewrite equation ([3.2](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.2)) as an equation for Dq in terms of t, q and p:

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-7.gif

Equations ([3.7](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.7)) and ([3.8](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.8)) give the rate of change of q and p along realizable paths as functions of t, q, and p along the paths.

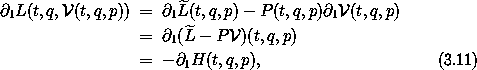
Define the function

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-8.gif

which is the Lagrangian reexpressed as a function of time, coordinates, and momenta.[2](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html" \l "footnote_Temp_224) For the equations of motion we need https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif1 L evaluated with the appropriate arguments. Consider

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-9.gif

where we used the chain rule in the first step and the inverse property ([3.5](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.5)) of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gifin the second step. Introducing the momentum selector[3](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html" \l "footnote_Temp_225) P(t, q, p) = p, and using the property https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif1 P = 0, we have



where the Hamiltonian H is defined by[4](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html" \l "footnote_Temp_226)

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-11.gif

Using the algebraic result ([3.11](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.11)), the Lagrange equation ([3.7](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.7)) for Dp becomes

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-12.gif

The equation for Dq can also be written in terms of H. Consider

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-13.gif

To carry out the derivative of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-49.gifwe write it out in terms of L:

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-14.gif

again using the inverse property ([3.5](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.5)) of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gif. So, putting equations ([3.14](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.14)) and ([3.15](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.15)) together, we obtain

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-15.gif

Using the algebraic result ([3.16](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.16)), equation ([3.8](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.8)) for Dq becomes

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-16.gif

Equations ([3.13](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.13)) and ([3.17](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.17)) give the derivatives of the coordinate and momentum path functions at each time in terms of the time, and the coordinates, and momenta at that time. These equations are known as Hamilton's equations:[5](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html" \l "footnote_Temp_227)

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-19.gif

Hamilton's equations are constructed from a real-valued function, the Hamiltonian. The Hamiltonian function is[6](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html" \l "footnote_Temp_228)

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-21.gif

The Hamiltonian has the same value as the energy function https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-45.gif(see equation [1.140](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-15.html#EQUATION_1.140)), except that the velocities are expressed in terms of time, coordinates, and momenta by https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gif:

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-22.gif

#### [Illustration](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html" \l "%_toc_%_sec_Temp_229)

Let's try something simple: the motion of a particle of mass m with potential energy V(x, y). A Lagrangian is

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To form the Hamiltonian we find the momenta p = https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif2 L(t, q, v): px = m vx and py = m vy. Solving for the velocities in terms of the momenta is easy here: vx = px/m and vy = py/m. The Hamiltonian is H(t, q, p) = pv - L(t, q, v), with v reexpressed in terms of (t, q, p):

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-24.gif

The kinetic energy is a homogeneous quadratic form in the velocities, so the energy is T + V and the Hamiltonian is the energy expressed in terms of momenta rather than velocities. Hamilton's equations for Dq are

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-25.gif

Note that these equations merely restate the relation between the momenta and the velocities. Hamilton's equations for Dp are

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-26.gif

The rate of change of the linear momentum is minus the gradient of the potential energy.

**Exercise 3.1.**  **Deriving Hamilton's equations**

For each of the following Lagrangians derive the Hamiltonian and Hamilton's equations. These problems are simple enough to do by hand.

**a**.  A Lagrangian for a planar pendulum: L(t, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-20.gif) = (1/2) m l2 https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-20.gif2 + m g l cos https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif.

**b**.  A Lagrangian for a particle of mass m with a two-dimensional potential energy: V(x, y) = (x2 + y2)/2 + x2 y - y3/3 is L(t; x, y; https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-37.gif, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-1.gif) = (1/2) m (https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-37.gif2 + https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-1.gif2) - V(x, y).

**c**.  A Lagrangian for a particle of mass m constrained to move on a sphere of radius R: L(t; https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gif; https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-20.gif, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-34.gif) = (1/2) m R2 (https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-20.gif2 + (https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-34.gif sin https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gif)2), where https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-19.gifis the colatitude and https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-16.gifis the longitude on the sphere.

**Exercise 3.2.**  **Sliding pendulum**

For the pendulum with a sliding support (see exercise [1.20](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-13.html#%_thm_1.20)), derive a Hamiltonian and Hamilton's equations.

#### [Hamiltonian state](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html" \l "%_toc_%_sec_Temp_232)

Given a coordinate path q and a Lagrangian L, the corresponding momentum path p is given by equation ([3.2](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.2)). Equation ([3.17](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.17)) expresses the same relationship in terms of the corresponding Hamiltonian H. That these relations are valid for any path, whether or not it is a realizable path, allows us to abstract to arbitrary velocity and momentum at a moment. At a moment, the momentum p for the state tuple ( t, q, v ) is p = https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif2 L(t, q, v). We also have v = https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif2 H(t, q, p). In the Lagrangian formulation the state of the system at a moment can be specified by the local state tuple ( t, q, v ) of time, generalized coordinates, and generalized velocities. Lagrange's equations determine a unique path emanating from this state. In the Hamiltonian formulation the state can be specified by the tuple ( t, q, p ) of time, generalized coordinates, and generalized momenta.

The Hamiltonian formulation and the Lagrangian formulation are equivalent.

Given a path q, the Lagrangian state path and the Hamiltonian state paths can be deduced from it. The Lagrangian state path https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-9.gif[q] can be constructed from a path q simply by taking derivatives. The Lagrangian state path satisfies:

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-27.gif

The Lagrangian state path is uniquely determined by the path q. The Hamiltonian state path https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-2.gifL[q] can also be constructed from the path q but the construction requires a Lagrangian. The Hamiltonian state path satisfies

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-28.gif

The Hamiltonian state tuple is not uniquely determined by the path q because it depends upon our choice of Lagrangian, which is not unique.

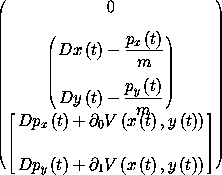
The 2n-dimensional space whose elements are labeled by the n generalized coordinates qi and the n generalized momenta pi is called the phase space. The components of the generalized coordinates and momenta are collectively called the phase-space components.[8](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html" \l "footnote_Temp_234) The dynamical state of the system is completely specified by the phase-space state tuple ( t, q, p ), given a Lagrangian or Hamiltonian to provide the map between velocities and momenta.

#### [Computing Hamilton's equations](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html" \l "%_toc_%_sec_Temp_235)

Hamilton's equations are a system of first-order ordinary differential equations. A procedural formulation of Lagrange's equations as a first-order system was presented in section [1.7](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-14.html#%_sec_1.7). The following formulation of Hamilton's equations is analogous:

The Hamiltonian state derivative is computed as follows:

Hamilton's equations are[9](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html" \l "footnote_Temp_236)



**Exercise 3.3.**  **Computing Hamilton's equations**

Check your answers to exercise [3.1](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#%_thm_3.1) with the Hamilton-equations procedure.

### [3. The Legendre Transformation](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html" \l "%_toc_%_sec_3.1.1)

The Legendre transformation abstracts a key part of the process of transforming from the Lagrangian to the Hamiltonian formulation of mechanics -- the replacement of functional dependence on generalized velocities with functional dependence on generalized momenta.

Given a real-valued function F, if we can find a real-valued function G such that DF = (DG)-1, then we say that F and G are related by a Legendre transform.

Locally, we can define the inverse function[11](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html" \l "footnote_Temp_239) https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gifof DF so that DF o https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gif= I, where I is the identity function I(w) = w. Consider the composite function https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-47.gif= F o https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gif. The derivative of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-47.gifis

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-30.gif

Using the product rule and DI = 1, we have

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-31.gif

or

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-32.gif

The integral is determined up to a constant of integration. If we define

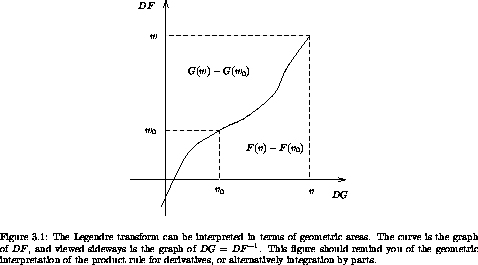
https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-33.gif

then we have

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The function G has the desired property that DG is the inverse function https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gifof DF. The derivation just given applies equally well if the arguments of F and G have multiple components.

Given a relation w = DF(v) for some given function F, then v = DG(w) for G = Ihttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gif - F o https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gif, where https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gifis the inverse function of DF provided it exists.



A picture may help (see figure [3.1](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#FIGURE_3.1)). The curve is the graph of the function DF. Turned sideways, it is also the graph of the function DG, because DG is the inverse function of DF. The integral of DF from v0 to v is F(v) - F(v0); this is the area below the curve from v0 to v. Likewise, the integral of DG from w0 to w is G(w) - G(w0); this is the area to the left of the curve from w0 to w. The union of these two regions has area w v - w0 v0. So

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which is the same as

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The left-hand side depends only on the point labeled by w and v and the right-hand side depends only on the point labeled by w0 and v0, so these must be constant, independent of the variable endpoints. So as the point is changed the combination G(w) + F(v) - wv is invariant. Thus

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with constant C. The requirement for G depends only on DG so we can choose to define G with C = 0.

#### 

#### [Legendre transformations with passive arguments](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_Temp_240)

Let F be a real-valued function of two arguments and

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If we can find a real-valued function G such that

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we say that F and G are related by a Legendre transformation, that the second argument in each function is active, and that the first argument is passive in the transformation.

If the function https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif1 F can be locally inverted with respect to the second argument we can define

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giving

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-42.gif

where W = I1 is the selector function for the second argument.

For the active arguments the derivation goes through as before. The first argument to F and G is just along for the ride -- it is a passive argument. Let

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then define

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-44.gif

We can check that G has the property https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gif= https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif1 G by carrying out the derivative:

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but

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-46.gif

or

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-47.gif

So

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as required. The active argument may have many components.

The partial derivatives with respect to the passive arguments are related in a remarkably simple way. Let's calculate the derivative https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif0 G in pieces. First,

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-49.gif

because https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif0 W = 0. To calculate https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif0 https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-47.gifwe must supply arguments:

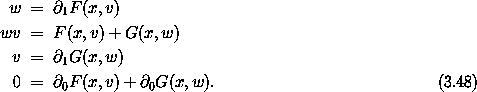
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Putting these together, we find

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The calculation is unchanged if the passive argument has many components.

We can write the Legendre transformation more symmetrically:



The last relation is not as trivial as it looks, because x enters the equations connecting w and v. With this symmetrical form, we see that the Legendre transform is its own inverse.

**Exercise 3.4.**  **Simple Legendre transforms**

For each of the following functions, find the function that is related to the given function by the Legendre transform on the indicated active argument. Show that the Legendre transform relations hold for your solution, including the relations among passive arguments, if any.

**a**.  F(x) = a x + b x2, with no passive arguments.

**b**.  F(x, y) = a sin x cos y, with x active.

**c**.  F(x, y, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-37.gif, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-1.gif) = x https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-37.gif2 + 3 https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-37.gifhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-1.gif+ y https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-1.gif2, with https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-37.gifand https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-1.gifactive.

#### [Hamilton's equations from the Legendre transformation](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html" \l "%_toc_%_sec_Temp_242)

The Lagrangian L and the Hamiltonian H are related by a Legendre transformation:

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https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-54.gif

and

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-55.gif

with passive equations

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-56.gif

Presuming it exists, we can define the inverse of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif2 L with respect to the last argument:

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-57.gif

and write the Hamiltonian

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-58.gif

These relations are purely algebraic in nature.

On a path q we have the momentum p:

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and from the definition of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gifwe find

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The Legendre transform gives

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-61.gif

This relation is purely algebraic and is valid for any path. The passive equation ([3.53](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.53)) gives

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but the left-hand side can be rewritten using the Lagrange equations, so

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This equation is valid only for realizable paths, because we used the Lagrange equations to derive it. Equations ([3.58](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.58)) and ([3.60](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.60)) are Hamilton's equations.

The remaining passive equation is

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-64.gif

This passive equation says that the Lagrangian has no explicit time dependence (https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif0 L = 0) if and only if the Hamiltonian has no explicit time dependence (https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif0 H = 0).

**Exercise 3.5.**

Using Hamilton's equations, show directly that the Hamiltonian is a conserved quantity if it has no explicit time dependence.

#### [Legendre transforms of quadratic functions](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html" \l "%_toc_%_sec_Temp_244)

More generally, we can compute a Legendre transformation for polynomial functions where the leading term is a quadratic form:

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-65.gif

We can assume M is symmetric,[12](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html" \l "footnote_Temp_245) because it defines a quadratic form. We can find linear expressions for w as

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So if M is invertible we can solve for v in terms of w. Thus we may define a function https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gifsuch that

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and we can use this to compute the value of the function G:

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#### [Computing Hamiltonians](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html" \l "%_toc_%_sec_Temp_246)

We implement the Legendre transform for quadratic functions by the procedure[13](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html" \l "footnote_Temp_247)

The procedure Legendre-transform takes a procedure of one argument and returns the procedure that is associated with it by the Legendre transform. If w = DF(v), wv = F(v) + G(w), and v = DG(w) specifies a one-argument Legendre transformation, then G is the function associated with F by the Legendre transform: G = I https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gif- F o https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gif, where https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.gifis the functional inverse of DF.

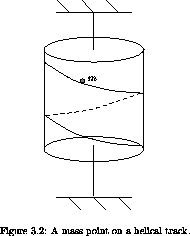
We can use the Legendre-transform procedure to compute a Hamiltonian from a Lagrangian:

For example, the Hamiltonian for the motion of the point mass with the potential energy V(x, y) may be computed from the Lagrangian:

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**Exercise 3.6.**  **On a helical track**

A uniform cylinder of mass M, radius R, and height h is mounted so as to rotate freely on a vertical axis. A mass point of mass m is constrained to move on a uniform frictionless helical track of pitch *ß* (measured in radians per meter of drop along the cylinder) mounted on the surface of the cylinder (see figure [3.2](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#FIGURE_3.2)). The mass is acted upon by standard gravity (g = 9.8 ms-2).



**a**.  What are the degrees of freedom of this system? Pick and describe a convenient set of generalized coordinates for this problem. Write a Lagrangian to describe the dynamical behavior. It may help to know that the moment of inertia of a cylinder around its axis is (1/2)MR2. You may find it easier to do the algebra if various constants are combined and represented as single symbols.

**b**.  Make a Hamiltonian for the system. Write Hamilton's equations for the system. Are there any conserved quantities?

**c**.  If we release the mass point at time t = 0 at the top of the track with zero initial speed and let it slide down, what is the motion of the system?

**Exercise 3.7.**  **An ellipsoidal bowl**

Consider a point particle of mass m constrained to move in a bowl and acted upon by a uniform gravitational acceleration g. The bowl is ellipsoidal, with height z = a x2 + b y2. Make a Hamiltonian for this system. Can you make any immediate deductions about this system?

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**Lecture 4**

### Lecture topic: [[Hamilton's Equations from the Action Principle](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html" \l "%_toc_%_chap_1)](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_3.1.2)

**The plan**

### [1. Hamilton's Equations from the Action Principle](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_3.1.2)

### [2. A Wiring Diagram](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_3.1.3)

## [3. Poisson Brackets](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_3.2)

### [1. Hamilton's Equations from the Action Principle](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_3.1.2)

The previous two derivations of Hamilton's equations have made use of the Lagrange equations. Hamilton's equations can also be derived directly from the action principle.

The action is the integral of the Lagrangian along a path:

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The action is stationary with respect to variations of the path that preserve the configuration at the endpoints (for Lagrangians that are functions of time, coordinates, and velocities).

We can rewrite the integrand in terms of the Hamiltonian

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with p(t) = https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif2 L(t, q(t), Dq(t)). The Legendre transformation construction gives

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which is one of Hamilton's equations, the one that does not depend on the path being a realizable path.

In order to vary the action we need to make the dependences on the path explicit. We introduce

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so

p(t) = https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-3.gif[q](t)

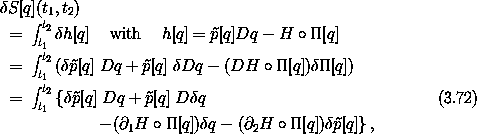
and

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The integrand of the action integral is then

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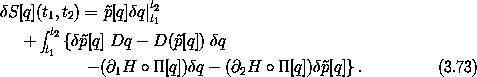
The variation of the action is



where https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-17.gifhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-3.gif[q] is the variation in the momentum.[15](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html" \l "footnote_Temp_251) Integrating the second term by parts, using

D(https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-3.gif[q] https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-17.gifq) = D(https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-3.gif[q]) https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-17.gifq + https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-3.gif[q] D https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-17.gifq,

we get



The variations are constrained so that https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-17.gifq(t1) = https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-17.gifq(t2) = 0, so the integrated part vanishes. The variation of the action is

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As a consequence of equation ([3.68](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.68)), the factor multiplying https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-17.gifhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-3.gif[q] is zero. We are left with

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For the variation of the action to be zero for arbitrary variations, except for the endpoint conditions, we must have

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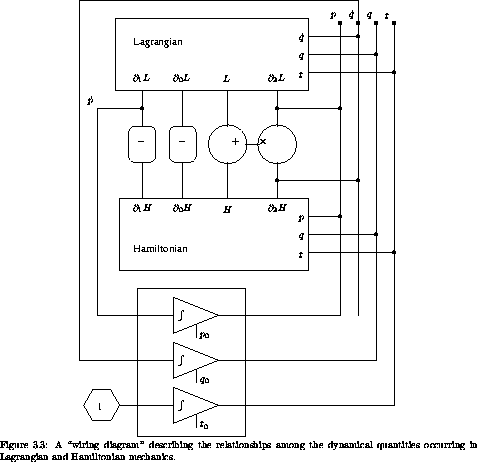
Using using p(t) = https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-3.gif[q](t), this is

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which is the ``dynamical'' Hamilton equation.

### [2. A Wiring Diagram](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html" \l "%_toc_%_sec_3.1.3)

Figure [3.3](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#FIGURE_3.3) shows a summary of the functional relationship between the Lagrangian and the Hamiltonian descriptions of a dynamical system. The diagram shows a ``circuit'' interconnecting some ``devices'' with ``wires.'' The devices represent the mathematical functions that relate the quantities on their terminals. The wires represent identifications of the quantities on the terminals that they connect. For example, there is a box that represents the Lagrangian function. Given values t, q, and https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-1.gif, the value of the Lagrangian L(t, q, https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-1.gif) is on the terminal labeled L, which is wired to an addend terminal of an adder. Other terminals of the Lagrangian carry the values of the partial derivatives of the Lagrangian function.



The upper part of the diagram summarizes the relationship of the Hamiltonian to the Lagrangian. For example, the sum of the values on the terminals L of the Lagrangian and H of the Hamiltonian is the product of the value on the https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-1.gifterminal of the Lagrangian and the value on the p terminal of the Hamiltonian. This is the active part of the Legendre transform. The passive variables are related by the corresponding partial derivatives being negations of each other. In the lower part of the diagram the equations of motion are indicated by the presence of the integrators, relating the dynamical quantities to their time derivatives.

One can use this diagram to help understand the underlying unity of the Lagrangian and Hamiltonian formulations of mechanics. Lagrange's equations are just the connection of the https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-4.gifwire to the https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif1 L terminal of the Lagrangian device. One of Hamilton's equations is just the connection of the https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-4.gifwire (through the negation device) to the https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif1 H terminal of the Hamiltonian device. The other is just the connection of the https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-1.gifwire to the https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif2 H terminal of the Hamiltonian device. We see that the two formulations are consistent. One does not have to abandon any part of the Lagrangian formulation to use the Hamiltonian formulation: there are deductions that can be made using both simultaneously.

P = I2.

The overall minus sign in the definition of the Hamiltonian is traditional.

In traditional notation, Hamilton's equations are written:

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or as separate equations for each component:

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-18.gif

Traditionally, the Hamiltonian is written

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-20.gif

This way of writing the Hamiltonian confuses the values of functions with the functions that generate them: both https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-1.gifand L must be reexpressed as functions of the time, coordinates, and momenta.

In the construction of the Lagrangian state derivative from the Lagrange equations we must solve for the highest-order derivative. The solution process requires the inversion of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif2 https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif2 L. In the construction of Hamilton's equations, the construction of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-27.giffrom the momentum state function https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif2 L requires the inverse of the same structure. If the Lagrangian formulation has singularities, they cannot be avoided by going to the Hamiltonian formulation.

The term phase space was introduced by Josiah Willard Gibbs in his formulation of statistical mechanics. The Hamiltonian plays a fundamental role in the Boltzmann-Gibbs formulation of statistical mechanics and in both the Heisenberg and Schrödinger approaches to quantum mechanics.

The momentum p can be viewed as the coordinate representation of a linear form on the tangent space. Thus phttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-1.gif is a scalar quantity that is invariant under time-independent coordinate transformations of the configuration space. The set of momentum forms comprise an n-dimensional vector space at each point of the configuration space called the cotangent space. The collection of all cotangent spaces of a configuration space forms a space called the cotangent bundle of the configuration manifold.

By default, literal functions map reals to reals; the default type for a literal function is (-> Real Real). Here, the potential energy V takes two real arguments and returns a real.

The Legendre transformation is more general than its use in mechanics in that it captures the relationship between conjugate variables in systems as diverse as thermodynamics, circuits, and field theory.

This can be done so long as the derivative is not zero.

If M is the matrix representation of M, then M = MT.

The division operation, denoted by / in the Legendre-transform procedure, is generic over mathematical objects. We interpret the division in the matrix representation as follows: a vector y divided by a matrix M is interpreted as a request to solve the linear system M x = y, where x is the unknown vector.

The function https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-2.gif[q] is the same as https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-2.gifL[q] introduced previously. Indeed, the Lagrangian is needed to define momentum in every case, but we are suppressing the dependency here because it does not matter in this argument.

The variation of the momentum https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-17.gifhttps://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-3.gif[q] need not be further expanded in this argument because it turns out that the factor multiplying it is zero.  However, it is handy to see how it is related to the variations in the coordinate path https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-17.gifq:

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-78.gif

It is sometimes asserted that the momenta have a different status in the Lagrangian and Hamiltonian formulations: that in the Hamiltonian framework the momenta are ``independent'' of the coordinates. From this it is argued that the variations https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-17.gifq and https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-17.gifp are arbitrary and independent, therefore implying that the factor multiplying each of them in the action integral ([3.74](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-37.html#EQUATION_3.74)) must independently be zero, apparently deriving both of Hamilton's equations. The argument is fallacious: we can write https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-17.gifp in terms of https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap1-Z-G-D-17.gifq.

## [3. Poisson Brackets](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_3.2)

Here we introduce the Poisson bracket, in terms of which Hamilton's equations have an elegant and symmetric expression. Consider a function F of time, coordinates, and momenta. The value of F along the path

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-5.gif(t) = ( t, q(t), p(t) ) is (F o https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-5.gif)(t) = F(t, q(t), p(t)).

The time derivative of F o https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-D-5.gifis

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If the phase-space path is a realizable path for a system with Hamiltonian H, then Dq and Dp can be reexpressed using Hamilton's equations:



where the Poisson bracket { F , H } of F and H is defined by[17](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-38.html" \l "footnote_Temp_253)

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Note that the Poisson bracket of two functions on the phase-state space is also a function on the phase-state space.

The coordinate selector Q = I1 is an example of a function on phase-state space:

Q(t, q, p) = q.

According to equation ([3.79](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-38.html#EQUATION_3.79)),

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but this is the same as Hamilton's equation

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Similarly, the momentum selector P = I2 is a function on phase-state space:

P(t, q, p) = p.

We have

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which is the same as Hamilton's other equation

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-92.gif

So the Poisson bracket provides a uniform way of writing Hamilton's equations:

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The Poisson bracket of any function with itself is zero, so we recover the conservation of energy for a system that has no explicit time dependence:

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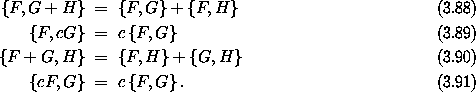
#### [Properties of the Poisson bracket](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html" \l "%_toc_%_sec_Temp_254)

Let F, G, and H be functions of time, position, and momentum, and let c be independent of position and momentum.

The Poisson bracket is antisymmetric:

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-95.gif

It is bilinear (linear in each argument):



The Poisson bracket satisfies Jacobi's identity:

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-97.gif

All but the last of ([3.87](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-38.html#EQUATION_3.87)-[3.92](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-38.html#EQUATION_3.92)) can immediately be verified from the definition. Jacobi's identity requires a little more effort to verify. We can use the computer to avoid some work. Define some literal phase-space functions of Hamiltonian type.

#### [Poisson brackets of conserved quantities](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html" \l "%_toc_%_sec_Temp_255)

The Poisson bracket of conserved quantities is conserved. Let F and G be time-independent phase-space state functions: https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif0 F = https://mitpress.mit.edu/sites/default/files/titles/content/sicm/front-Z-G-D-2.gif0 G = 0. If F and G are conserved by the evolution under H then

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So the Poisson brackets of F and G with H are zero: { F, H} = { G, H} = 0. The Jacobi identity then implies

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and thus

https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-100.gif

so { F, G } is a conserved quantity. The Poisson bracket of two conserved quantities is also a conserved quantity.

In traditional notation the Poisson bracket is written

*https://mitpress.mit.edu/sites/default/files/titles/content/sicm/chap3-Z-G-87.gif*

## [*One Degree of Freedom*](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-4.html#%_toc_%_sec_3.3)

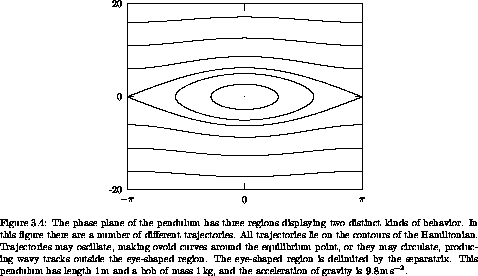
The solutions of time-independent systems with one degree of freedom can be found by quadrature. Such systems conserve the Hamiltonian: the Hamiltonian has a constant value on each realizable trajectory. We can use this constraint to eliminate the momentum in favor of the coordinate. Thus Hamilton's equations reduce to a single equation Dq(t) = f(q(t)). The solution q can be expressed as a definite integral.

A geometric view reveals more structure. Time-independent systems with one degree of freedom have a two-dimensional phase space. Energy is conserved, so all orbits are level curves of the Hamiltonian. The possible orbit types are restricted to curves that are contours of a real-valued function. The possible orbits are paths of constant altitude in the mountain range on the phase plane described by the Hamiltonian.

Only a few characteristic features are possible. There are points that are stable equilibria of the dynamical system. These are the peaks and pits of the Hamiltonian mountain range. These equilibria are stable in the sense that neighboring trajectories on nearby contours stay close to the equilibrium point. There are orbits that trace simple closed curves on contours that surround a peak or pit, or perhaps several peaks. There are also trajectories lying on contours that cross at a saddle point. The crossing point is an unstable equilibrium, unstable in the sense that neighboring trajectories leave the vicinity of the equilibrium point. Such contours that cross at saddle points are called separatrices (singular: separatrix), contours that ``separate'' two regions of distinct behavior.

At every point Hamilton's equations give a unique rate of evolution and direct the system to move perpendicular to the gradient of the Hamiltonian. At the peaks, pits, and saddle points, the gradient of the Hamiltonian is zero, so according to Hamilton's equations these are equilibria. At other points, the gradient of the Hamiltonian is nonzero, so according to Hamilton's equations the rate of evolution is nonzero. Trajectories evolve along the contours of the Hamiltonian. Trajectories on simple closed contours periodically trace the contour. At a saddle point, contours cross. The gradient of the Hamiltonian is zero at the saddle point, so a system started at the saddle point does not leave the saddle point. On the separatrix away from the saddle point the gradient of the Hamiltonian is not zero, so trajectories evolve along the contour. Trajectories on the separatrix are asymptotic forward or backward in time to a saddle point. Going forward or backward in time, such trajectories forever approach an unstable equilibrium but never reach it. If the phase space is bounded, asymptotic trajectories that lie on contours of a smooth Hamiltonian are always asymptotic to unstable equilibria at both ends (but they may be different equilibria).

These orbit types are all illustrated by the prototypical phase plane of the pendulum (see figure [3.4](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-39.html#FIGURE_3.4)). The solutions lie on contours of the Hamiltonian. There are three regions of the phase plane; in each the motion is qualitatively different. In the central region the pendulum oscillates; above this there is a region in which the pendulum circulates in one direction; below the oscillation region the pendulum circulates in the other direction. In the center of the oscillation region there is a stable equilibrium, at which the pendulum is hanging motionless. At the boundaries between these regions, the pendulum is asymptotic to the unstable equilibrium, at which the pendulum is standing upright.[18](https://mitpress.mit.edu/sites/default/files/titles/content/sicm/book-Z-H-39.html" \l "footnote_Temp_256) There are two asymptotic trajectories, corresponding to the two ways the equilibrium can be approached. Each of these is also asymptotic to the unstable equilibrium going backward in time.



The pendulum has only one unstable equilibrium. Remember that the coordinate is an angle.

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**Lecture 5**

**Lecture topic: Work and Energy**

**The plan**

**1. Work**

**2. Energy**

2.1. Kinetic Energy

2.2. Potential Energy

**3. Energy is neither gained nor lost in any process**

**4. Using the Law of Conservation of Energy**

**1. Work**

*Work is done when a force causes an object to move.*

To be specific, work is the product of the component of a force in the direction of the displacement it causes and the magnitude of the displacement. For work to be accomplished a force has to move an object and the force and displacement have to be in the same direction. If you lift a book a distance of one meter, you have done work on it. The book moved up and the force was directed up. If you hold a heavy book stationary at some height above the floor, you have done no work on it. The force was up, but there was no displacement, so there was no work.

A simple equation for work is:

.

***W*** is work, ***F*** is the applied force, and ***Δr*** is the change in displacement.

This is a “special case” equation. The force and the displacement have to be in the same direction. If you took physics last year, this was the equation that you used.

* You lift a 1.2 kg book a distance of 1.0 m, how much work did you do on the book?

This is a simple problem. The force doing the work has the same magnitude as the weight of the book (that’s what it takes to lift it), ***mg***.

.

The unit we ended up with is a Newton-meter. This is defined in physics as a joule. The symbol for the joule is *J*. (The joule is named after James Joule, a big name in the area of energy.)

.

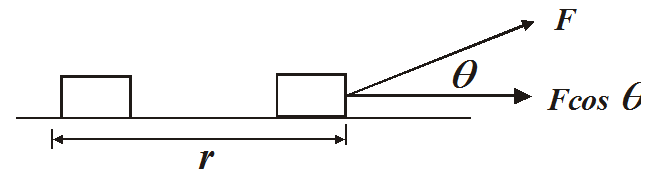
In the United States the most common unit of work is the foot pound.

What happens when the force and the displacement aren’t in the same direction?

In mathematical terms, we get this general equation:

.

Here the ‘***F****’* and ‘***cos θ*** ‘ part of the equation is the component of the applied force that is in the direction of the displacement



The drawing above represents the motion of a crate being moved by an applied force. The crate is moved a distance . The work done is simply

.

We find this equation in its AP form as:

.

Here is work, is the net force, is the distance the object is moved, and  is the angle the net force makes with the direction of motion.

It’s two equations in one. The first one is for when the angle is zero and the net force has the same direction as the motion. So we can say that 

The second one is used when there is an angle between the net force and the motion direction.

* A force is applied to a crate at an angle of 25°. The crate is dragged across the deck a distance of 2.5 m. If the amount of work done after it has been moved is 1 210 J, what was the applied force?

This is a simple problem, all we have to do is solve it for ***F***!

.

Work is a scalar quantity. This means you can add and subtract work without treating it as a vector. How convenient!

Joules are a small unit, so it is very common to deal with kJ and MJ. A joule is roughly the amount of work you do when you lift a Big Mac one meter. That’s not very much work, is it?

This brings us to the next exciting topic: energy.

**2. Energy**

Energy is another one of those common terms that you hear all the time. Interestingly enough, in everyday language, it is used pretty much correctly. Imagine that.

In physics we define energy as:

***Energy ≡ the ability to do work.***

But what does that mean? Well work is done when something is displaced by a force. If work is done, it takes energy. If you lift a 1 N rock 1 m, you’ve done 1 J of work and expended 1 J of energy.

Energy and work, intimately related as they are, use the same unit.

Energy comes in a vast array of types and all sorts of ways have been devised to classify the different types of energy. No doubt you can think of lots of them. There’s electrical energy, solar energy, nuclear energy, thermal energy, chemical energy, etc. lots of energy types.

Initially we will be dealing with mechanical energy. This is the energy associated with motion and forces. There are two types of mechanical energy (these types can be applied - and often are – to all types of energy, keep in mind). The two types are ***kinetic energy*** and ***potential energy***.

***2.1. Kinetic Energy***

This is the energy of motion. When a system is moving, it has kinetic energy. Thus the object’s motion can be transformed into work. All this means is that a moving object can hit something and make it move, thus accomplishing work. The unit for kinetic energy is the joule. The equation for kinetic energy is:

.

***K*** stands for kinetic energy, ***m*** is the mass of the object, and ***v*** is its velocity.

* How much kinetic energy does a 2.5 kg ball possess if it is thrown with a velocity of 21 m/s?

Very simple problem, just use the equation.

.

Can you see how the unit ends up being a joule?

When an object is given kinetic energy (by applying a force to it, of course), the amount of work required to do this is, ignoring friction, equal to the kinetic energy the object gains.

We can say this in a more fancy way. We say that the net work done on an object by a net force is equal to the change in its kinetic energy.

.

If it starts at rest, then the ***K0*** term is zero and the net work is equal to the final kinetic energy.

* A net 6 500 N force is applied to a resting 1 500 kg car, moving it forward. What is its kinetic energy and speed after being displaced 150 m?

There are two equations we can use:

.

The car begins at rest and the force is in the direction of the car’s motion. And

.

First we calculate the amount of work the force does, this will give us the amount of kinetic energy the car ends up with.

.

This is equal to the kinetic energy of the car at the end of the event. So

.

Now to find the velocity. We can use the equation for kinetic energy, since we’ve calculated its value:

.

Anything in motion has kinetic energy – falling water, the wind, moving electrons, etc. We will find that analyzing an object’s kinetic energy is a very powerful weapon in the old physics problem-solving arsenal.

***2.2. Potential Energy***

Potential energy is stored energy. There are many ways that energy can be stored. An electric battery represents stored energy, water piled up behind a dam, and chemical bonds in molecules to name just a few are also common ways to store energy. The type of potential energy that we will initially be interested in has to do with the energy of position brought about by gravity and by a spring being compressed or elongated.

First let’s look at gravitational potential energy. When an object is lifted to a height above a reference frame, work is done and object gains potential energy. The energy it gains is equal (ignoring friction) to the work done on it. The standard form that the equation for potential energy of position takes is:

,

although it’s usually written as simply:

.

***U*** is potential energy, ***m*** is the mass, ***g*** is the acceleration of gravity, and ***y*** is the vertical displacement.

It is also very common to write it in a slightly different form as:

,

where (standing for height) simply means the vertical displacement.

An object which is lifted to some new position can, if released from that position, do work as it falls back down. Old-fashioned clocks use weights in this way to power the clockwork mechanism.

The net work done by falling object is simply the change in potential energy.

.

When solving potential energy problems, the reference frame should be chosen to simplify the solution.

One sets the bottom position as zero and then all other displacements are measured in reference to the zero position.

* A 12 500 kg boulder is 157 m above a cabin. What is its PE with respect to the cabin?

,

.

***Conservation of Energy:*** One of the most important laws in all of science is the ***law of conservation of energy***. In chemistry you probably looked at it in this form: energy cannot be created or destroyed.

**3. Energy is neither gained nor lost in any process**

Energy can be transformed from one type to another, but, in any closed system the amount of energy cannot change.

We can examine conservation of energy by dropping a rubber ball. The ball begins at some initial height – it therefore has a certain amount of potential energy. As it is not moving, it has no kinetic energy whatsoever. The ball is released and starts falling downward. As it falls it accelerates and falls faster and faster. This means that its kinetic energy is increasing as it falls. Its potential energy is decreasing because its height is becoming smaller. What is happening is that its potential energy is being converted into kinetic energy. Just before it hits, all of its potential energy will have been transformed into kinetic energy.

But what happens when it hits?

The ball deforms on contact; it gets squished. Its kinetic energy is being transformed into potential energy – the same kind of potential energy as you would find in a compressed spring or stretched rubber band. The ball then “unsquishes” itself and rebounds. The potential energy stored in the ball is transformed back into kinetic energy and the ball goes back up. On the bounce, this kinetic energy is converted back into potential energy as the ball moves upward. When the ball stops, all of its kinetic energy has been transformed into potential energy, and so on.

The interesting thing, and a point that makes one doubt the law of conservation of energy, is that the ball doesn’t bounce back to its original height. So the uninformed observer thinks, “Hey, energy got lost here! It didn’t go to the same height so some of its energy was destroyed! So much for the stupid law of conservation of energy.”

Well, the law of conservation of energy ***always*** works – it ***is*** the ***law***, after all. What happens is that the energy of the ball is transformed into energy forms that do not contribute to the bounce height. We call these transformations ***energy losses*.** They are not really energy losses, however, in the sense that energy has been destroyed.

When the ball falls it encounters friction with the air. It must push aside air molecules, giving them some of its energy. The air molecules gain energy during the collisions with the ball as it falls and some of the molecules making up the ball also gain energy. The effect of this is to heat the air and the ball to slightly higher temperatures. This means that its kinetic energy is less than what is expected (but not by much, the decrease in energy for a ball falling a few meters on the surface of the earth is almost insignificant). The ball finally hits the deck. Most of its kinetic energy goes into making the ball deform. When the ball “undeforms itself” this energy is converted back into kinetic energy since the ball will be going up. The deformation, however, causes the individual molecules that make up the ball to gain energy - they vibrate more; the effect of this is to heat the ball. The energy that makes the ball’s particles vibrate more vigorously is no longer available as potential energy in the squished ball. The ball’s temperature increases. This means that the potential energy stored in the ball is less than the kinetic energy it began with before the squish. The deck under the ball is also distorted and heated slightly.

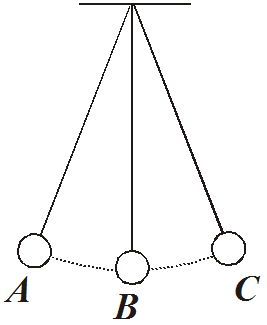
The ball also produces a noise when it hits, this is another energy loss. Energy is also lost to friction with the air as it rises after the bounce. The effect of these energy losses is that the ball doesn’t bounce as high as the place from whence it came.

If we total up the sum of all the energies and energy transformations that occurred during the ball drop event (the frictional heating, the deformational heating, the sound produced, etc.), however, we would find that the final sum of energy is equal to the initial amount of energy.

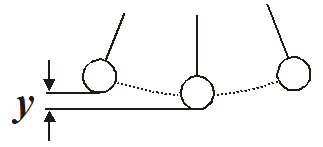
One of the demonstrations we did was one with a bowling ball, the one where it swung back and forth. Remember? A bowling ball was hung from a line that was secured to a hook in the ceiling. The ball was swung like a big pendulum.

Looking at the drawing of the ball’s path to the right, where does the ball have maximum/minimum kinetic energy and where does it have maximum/minimum potential energy?

Point ***A*** represents the maximum deflection to the left, point ***B*** represents the lowest point in the path, and point ***C*** represents the maximum deflection to the right.



At point ***A*** the ball is held at rest. All of its energy is potential energy because of its height ***y*** above the lowest point of travel.



The ball is released and its potential energy is converted into kinetic energy. At point B, the lowest point in its path, all of the energy will be kinetic energy. With respect to point B, chosen as our zero point, it has no potential energy (although it does have potential energy with respect to the deck below, but the rope won’t let it fall). As it rises to reach point C the kinetic energy is converted into potential energy. At point C all of its energy is potential. As the ball swings back and forth, the energy is transformed continually from potential energy to kinetic energy and so on.

Of course energy is converted into other forms besides potential and kinetic energy, so the ball will eventually come to rest. Each swing of the ball will end up at a slightly lower position until the ball stops moving. This is why the brave student could release the ball from a position where it was touching her nose and be absolutely confident that it would not come back and smash in her teeth.

The ball had to obey that laws of physics, didn’t it.

The law of conservation of energy was developed during the 1800’s. It is credited to Hermann von Helmholtz, although most of the work was done by James Joule. Turned out that nobody believed Joule (he was of the lower class and did not have a great deal of prestige at the time) but von Helmholtz had a grand reputation – he had the “von” thing going with his name, see, and so physicists believed him and gave him the credit. Joule got the final victory, however, there is no unit named the”von Helmholtz”.

Anyway, the implication of the law is really profound. It’s easy to accept that the energy of a small isolated system is constant, but what about the universe? The universe is an isolated system, ain’t it? You bet it is. This means that the total energy within the universe is unchanging. The universe began with “***x***” amount of energy around 15 billion years ago and today, it still has that same amount of energy.

This is, if you think about it, mind-boggling.

This means that for 15 billion years the universe’s energy has been busy plugging away changing from one form to another. Will it ever get tired of doing this?

**4. Using the Law of Conservation of Energy**

The energy in an isolated system can’t change. This is powerful stuff.

*Energy Before = Energy After*

or

*E* = *E'.*

(The little apostrophe mark added to the *E* making it *E'* means the quantity after some event. We pronounce *E’* as “E prime”.)

Let us look at a simple example. A rock is held at some height ***y*** above the deck. The rock is dropped and falls. We examine the energy before and the energy after and, thanks to the law, determine that the two quantities must equal each other. Let us note that we are ignoring all other energy losses, which is reasonable for this sort of event. Very little energy is lost by the rock as it falls a few meters.

The energy before (prior to being dropped) is:

.

The energy after (just before the rock hits) is:

.

Using these two relationships, we can write a general equation for the example

.

In general, without knowing the specifics, we can write the following equation:

.

This simply means that the energy before is the sum of its initial kinetic energy and potential energy. The energy after the event is the sum of its final kinetic and potential energy.

* A 1.5 kg ball is dropped from a height of 2.3 m. What is its speed the instant before it strikes the deck?

This problem could be solved using the acceleration equations that we’ve already learned about, but it can also be solved quite easily using the law of conservation of energy.

We can assume that the ball’s potential energy at the top will equal its kinetic energy just before it hits.

Thus,

.

The ball has no initial kinetic energy and no final potential energy, so the equation can be simplified by dropping out their terms. We can also eliminate the little subscript thingees.



Solve for the speed. Also note that the masses cancel out on each side.

.

A 500.0 g ball is thrown straight upward with a velocity of 35.4 m/s. How high does it go?

Use the law of conservation of energy. Rather than write out all the terms of the energy equation, we’ll just use the parts that apply.



solve for ***y***.

.

These problems are quite simple. The conservation of energy can also be used to solve some really complicated problems.

A roller coaster train is at rest at the top of a hill, the brakes are released and it rolls down some sort of a curved slope. What will be its speed at the bottom of the hill?

This seems to be a very complicated problem. The train will not be accelerating at  because it’s going down a ramp. It isn’t even a straight ramp, they usually curve, so how can you find its speed?

Conservation of energy! The energy at the top equals the energy at the bottom! (Ignoring friction of course.)

1. A roller coaster pauses at the top of a 75 m hill. What will be its speed at the bottom of the hill?

.

1. The first hill on a roller coaster is 94 m tall, the second hill is 68 m tall. If it starts from rest on the first hill, what theoretical speed will the roller coaster have on the second hill?

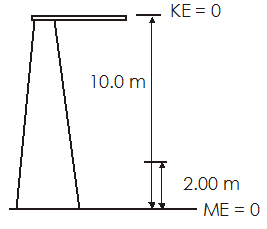


The train has only potential energy at the top of the first hill, but it has both potential and kinetic energy at the top of the second hill. This is because it is moving and is not at the bottom of the system at zero vertical displacement.

,

.

Let’s complicate things a bit more.



* A 655 N diver leaps into the water from a height of 10.0 m. What is his speed 2.00 m above the water?

,

,

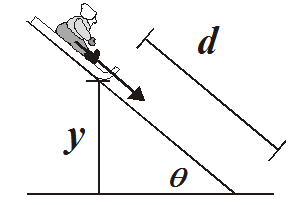
.

An even simpler method was to have your zero displacement position be 2.00 meters above the surface of the water, then the energy equation would be:

.

* A sled and rider together have a mass of 87 kg. They are atop a hill elevated at 42.5°. They slide down the slope a distance of 35 m and reach the bottom. Find the speed at the bottom of hill. Assume no friction.

The simplest way to solve the problem is to use conservation of energy.



.

We need to calculate the height.

,

,

.

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**Lecture 6**

**Lecture topic: Hooke's law in spring compression. Work and Displacement**

**The plan**

1. Hooke's law in spring compression

2. Work and Displacement

3. Examples

**1. Hooke's law in spring compression**

Another major potential energy area is the spring (the wire coil deals). Now you’ve seen springs before and know what they do. You squeeze a spring, i.e., compress it, and it wants to get back to its original length, so it can exert a force once it’s been compressed. See, it wants to “spring” back.

Compressed springs store energy.

The force to compress spring is proportional to the displacement. Mathematically:

,

where is the force, is displacement, and is known as the spring constant.

The spring constant is unique for each spring. The spring constant has units of .

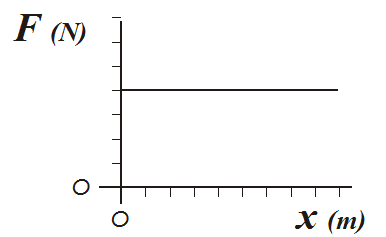
The amount of force it takes to compress a spring is equal to the force the compressed spring can exert. This is known as Hook’s law.

 - ***Hooke’s Law***

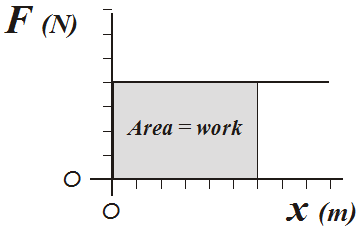
The minus sign simply means that the direction for the spring’s force is opposite of the direction of the force that compressed it. This makes sense right? If you compress a spring to the right, it will exert a force to the left. That’s all the minus sign means.

**2. Work and Displacement**

The work done by a system can be easily found by analyzing a graph of force vs displacement.



Here is a simple graph for constant force acting over a distance. The work done is simply the area under the curve of the force/displacement graph

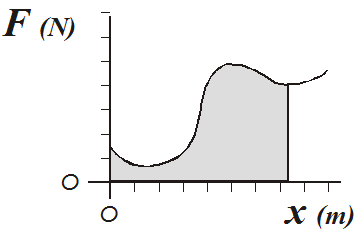


The work done by a 4.0 N force acting over a distance of 6.0 m would be the rectangular area delineated on the graph below. Since it is a rectangle, the work would be the height times the width or ***F*** multiplied by ***x.***  This is the equation we’ve been using for work. So the work would be 24 J.

Making a graph seems like an awful lot of trouble when we’ve got a wonderfully simple equation that we can use. But what about if the force isn’t constant?

Well, it turns out that the area under the curve is ***always*** going to be the work. What happens with the force isn’t all that important.

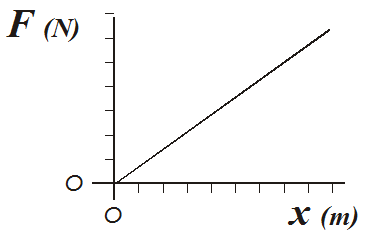
Here’s a graph of force vs displacement.



On this graph we have a pretty complicated curve. The work done in going from zero displacement to 7.3 m is still the area under the curve. Figuring out the area for this example would be a lot of trouble, basically you’d have to do it graphically or else use integral calculus – which most of you won’t have studied yet.

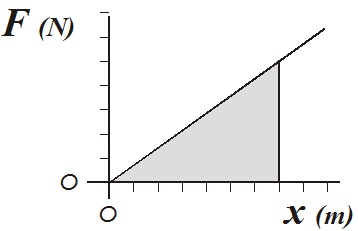
It can be simple though if you get regular geometric shapes. For example let us graph force vs displacement for a spring. The curve is a straight line, the y intercept is zero and the slope is the spring constant.

Here is just such a graph:



The work done in compressing the spring a certain distance is simply the area under the curve. We also know that the work done on the spring will equal the potential energy stored in the spring. So by analyzing the graph we can come up with an equation for the potential energy stored in a spring.

The area is a right triangle. From geometry we know that the area of such a triangle is simply:



.

The base of the triangle is simply ***x***. The height is ***F***.

But the force is a function of the displacement:



Plug this value for ***F*** into the area equation along with ***x*** as the base and you get:

.

This means that the potential energy of a spring is:

.

This equation will be available to you on the test.

**3. Examples**

* ***Example 1*.** A spring has a constant of 125 N/m. If the spring is compressed a distance of 13 cm, what is (a) the force required to do this and (b) the potential energy stored in the spring?

(a) To find the force, we use the equation for force and displacement:

.

Note that we aren’t using a minus sign. This is because we know the direction of the force and a minus sign would be extra trouble. We simply can’t be bothered.

(b) To find the potential energy, we use the potential energy equation for a spring:

.

We can now add the potential energy of a spring to the conservation of mechanical energy equation:

,

.

This equation deals with changes in energy for: kinetic energy, gravitational potential energy, and potential energy of a spring.

We can now use this to solve the odd problem or two. This is called, “applying the theory.” Sadly, it will be pretty dry – but go through it anyway, it’ll be good for you.

* ***Example 2.*** A 0.450 kg block, resting on a frictionless surface is pushed 8.00 cm into a light spring, ***k*** = 111 N/m. It is then released. What is the velocity of the block as it just leaves the spring?

We can solve this by analyzing the energy situation. There are only two terms we need worry about, the potential energy stored in the spring and the kinetic energy when the spring uncoils and the block is released. There is no initial kinetic energy and there is no final energy left in the spring. There is no change in gravitational potential energy. So we can develop the following equation:



the  terms cancel out

,

.

* ***Example 3.*** A 255 g block is traveling along a smooth surface with a velocity of 12.5 m/s. It runs head on into a spring (***k*** = 125 N/m). How far is the spring compressed?

.

* ***Example 4.*** A 1.0 kg ball is launched from a spring (k = 135 N/m) that has been compressed a distance of 25 cm. The ball is launched horizontally by the spring, which is 2.0 m above the deck.
* (a) What is the velocity of the ball just after it leaves the spring?
* (b) What is the horizontal distance the ball travels before it hits?
* (c) What is the kinetic energy of the ball just before it hits?

(a) ,

(b) Find the time to fall: ,

.

(c) To directly calculate the kinetic energy of the ball, we would have to calculate what it’s velocity is just before it hits. This would be a complicated problem – vectors, ***x*** and ***y*** components etc. Much easier to calculate it using conservation of energy. Its energy at the top, which will be the potential energy in the spring plus the gravitational potential energy because of the ball’s height, has to equal its kinetic energy at the bottom. We could also use its kinetic energy plus its gravitational potential energy at the top just before it leaves the table (this is because its kinetic energy must equal the potential energy stored in the spring before launch).

,

.

* ***Example 5.*** A spring is compressed a distance of 12 cm by a 675 g ball. The spring constant is 225 N/m. The ball is on a smooth surface as shown. The spring is released and sets the ball into motion.
* Find:

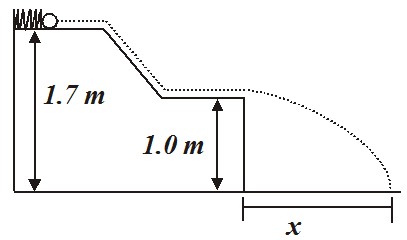
(a) the speed of the ball when it leaves the spring,

(b) the speed of the ball just before it leaves the edge of the table,

(c) the kinetic energy of the ball just before it hits the deck,

(d) the horizontal distance, ***x***, it travels when it leaves the table until it hits the deck.

Assume that the ball rolls down the ramp.



1. ,
2. ,

,

(c) ,

(d) ,

 .

* ***Example 6.*** A 49 kg projectile has a kinetic energy of 825 kJ when it is fired at an angle of 23.0°.
* Find

(a) the time of flight for the projectile and

(b) the maximum range of the projectile.

We need to know the initial velocity using its kinetic energy:

,

**,

**

1. Find the time:

,

1. .

* ***Example 7.*** A 25 kg box is pushed along a horizontal smooth surface with a force of 35 N for 1.5 m. It then slides into a spring (*k* = 1 500 N/m) and compresses it.
* Find the distance that the spring is compressed.

Find the acceleration:

.

Find the final velocity

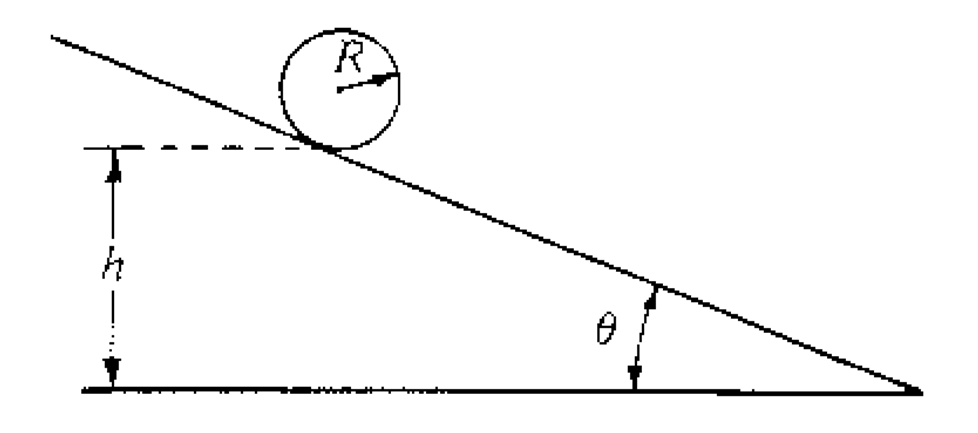
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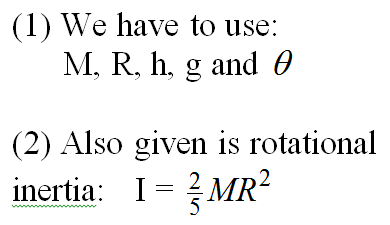
Now we can find the kinetic energy of the thing:



We can now find the displacement of the spring using the kinetic energy we just found:

• ***Example 8.*** Here's the 'rolling without slipping' problem. (static and conservation of 'E')





(a) Ei = Ef (no sliding or kinetic friction to generate heat or loss of mechanical energy)

mgh = KT + KR

(note the 2 types of kinetic energy: Translational and Rotational),

Mgh = .

Now cancel out the M and R2 and...

gh = .

Now solve for v2:

v2 = .

(i) KT = ½ M= .

(ii) KR == .

(b) On the inclined plane (no slipping): Assuming constant acceleration down the plane...

(i) .

 <

(from our Inclined Planed Theorem)

(ii) Using the center of the sphere as our 'pivot point':

I =  and ,

 and at = ,

.

(c) E = .

(d) First of all, the hollow sphere (arms out) has a greater I (I recall it's ).

For the same torque, we have an 'alpha' less than that of the solid sphere:  but wait! Let's do the math... KR =  and from above

,

KR = .

Method 2: Work = & Work =  (same torque, same) ... 

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**Lecture 7**

**Lecture topic: Power. Variant Power Equations**

**The plan**

**1. Power**

**2. Variant Power Equations**

**3. Examples**

3.1. Example about the Sojourner rover vehicle

**1. Power**

Power is simply the rate at which you do work. Do work fast, you get a lot of work done, and you are very powerful. If your power output is small, however, it takes a lot of time to do the work.

In Medieval times, oxen were used to do work – plow fields, that sort of thing. Oxen could do the work in a certain amount of time and generated a certain amount of power. Horses could work faster, but, unfortunately couldn’t be hitched to a plow. The hitching equipment developed for use with oxen would tend to strangle the unfortunate horse.

The way this worked was that the load from the equipment (plow) was transferred to the oxen with a wooden beam across the animal’s chest. This worked great for an ox’s anatomy, but not so good on a horse. It turns out that the wooden beam would squish a horse’s trachea, cutting off the poor critter’s wind. This tended to decrease the amount of work that got accomplished if not outright killing the poor beast. So horses were used to pull light wagons or for riding, but were not put to use by farmers to do heavy work. Then a wonderful invention came along, the horse collar. This was a semicircular device that went around the horse’s neck and distributed the load to the animal’s shoulders.

The critter’s neck came out through the circular part. This meant that horses could be used to do real, heavy work like plowing fields. Horses are more powerful than oxen, so the work could be done faster, a farmer could cultivate a greater area, and grow more food. More food in less time, quite a deal, in fact it was so significant that it caused enormous social change, leading to the modern world that we know.

The equation for power is:

.

You end up with a unit of a joule divided by a second. Power is given its own unit, the ***watt***. The symbol for the watt is ***W***. A watt is a joule per second.

.

* ***Example.*** It takes an engine 25 seconds to do 1 700 J of work.
* How much power did it develop?

This is a really simple problem; it’s just a plug n’ chug deal.

.

The watt is named after James Watt, a British engineer who perfected the steam engine. Watt himself developed his own unit for power, one that is still in use. This is the beloved ***horsepower***. The symbol for horsepower is ***hp***.

Both of these units are in common use in the United States of A. Horsepower is used for cars and larger motors. The Physics Kahuna’s Memsahib’s vacuum cleaner is advertised to be a 5.6 hp machine. Automobile engines are typically rated between 50 hp to 350 hp. Racecar engines produce more power, F1 race engines produce over a thousand horsepower.

The watt is often used for appliances that work off electricity – toasters, refrigerators, waffle irons, coffee makers, heaters, etc.

Here is the conversion factor for horsepower and watts.

.

**2. Variant Power Equations**

The equation for power is:

.

This is the form of the equation given on the physics. We will usually drop the “*avg”* bit and just use:

.

Recall that there are several different equations for work:

.

One finds the work using these equations and then divides by time to find the power.

One equation that can be developed is:

.

Of course you immediately recognized that the  term is simply the velocity of the system. Thus, substituting velocity for this term, we get:

.

We know that the velocity and force must be in the same direction. If they are separated by some angle *θ*, we get:

.

These are put together into another equation for the test:

.

This is the second power equation that you will be supplied when you take the test.

***General:*** Power depends on work and time. The faster the work is done, the greater the power that was developed. Work doesn't care about time, it only cares about a force acting, however slowly or quickly, to bring about the movement of an object.

A powerful engine is able to do work faster than a less powerful engine. They may well do the same amount of work, but the more powerful machine will do the work much more quickly.

**3. Examples**

* ***Example 1.***A human fly climbs up the outside of tall building to thrill the teeming hordes of earthlings below who fear he will fall to his doom. So if the 52 kg human fly takes 18 minutes to climb a 350 m building/
* How much power did he develop in the climb?

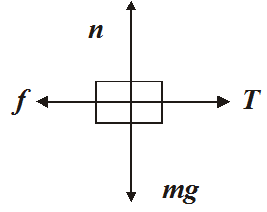
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* ***Example 2.***A 47 kg bicycle rider develops 0.26 hp. She rides the Featherlite 250 which has a mass of 2.3 kg. Anyway, the rider must climb a 235 m hill.
* How much time will this take?

, ,

.

* ***Example 3.***A 15.5 kg block is pulled across a flat deck at a constant speed of 3.0 m/s with a rope. The rope is horizontal to the deck. The coefficient of kinetic friction is 0.330.
* How much power does it take to do this?



We can find the force of friction, once that is found we can then calculate the force.

Since the block is moving at a constant speed, the sum of the horizontal forces must be zero.

Also since it isn’t falling or rising, the sum of the vertical forces must be zero.

Up is positive as is going to the right.

***x*** – direction:

.

***y*** – direction:

.

By definition, the frictional force is:

.

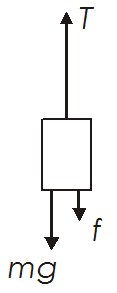
Substitute in the value for ***f***:

.

Now we can find the power it takes to do this:

.

* ***Example 4.*** A 1250 kg Elevator carries a maximum load of 955 kg. A constant frictional force of 3850 N exists.
* What minimum power (in hp) for the motor is needed to lift the thing at a constant speed of 3.50 m/s?



We need to find the force needed to lift the elevator. We draw a FBD and examine the sum of the forces. They must equal zero since the elevator will move at a constant speed.

,

,



the force is *T*, so:

,

.

***Example 5.*** Restoring spring force: F = -kx3 (no friction, horizontal surface)

Answer in terms of spring constant, k, amplitude, A, and mass, M.

(a) 

So to get to potential energy (U) from force (F), we have to integrate:

.

To get rid of the integration constant C notice that F and therefore U both equal zero when x = 0, so C = 0.

U = ¼ kx4 and x = A when released, so .

(b) Maximum speed (and K.E.) will occur when U=0 (when x = 0).

K = ¼ kx4 = ½ mv2 .

(c) Displacement (or x) when

U = K = ½ E = ½()

,

.

Now increase the amplitude. How is the period, T, affected? Recall that for a Hooke's Law spring,

T = ,

the period of oscillation was 'amplitude independent'.

(d) F = ma gives us:

-kx3 = ,

Hooke's Law spring:

–kx = ,

t = .

Let's integrate with our calculator from x = 0 to A = 5 just to see if the time changes when A = 10...

t =  = ?

½ mv2 + ¼ kx4 = ¼ kA4 (K + U = ETotal).

Solving for v:

v = ,

period

T = 

Now for some BC Calculus!

Let

,

T = ,

then



Since the integral is independent of A

.



Except for the negative sign, both methods give:

T value of .49942 sec for A = 5 and  and half as much for A = 10.

**3.1. Example about the Sojourner rover vehicle**

The Sojourner rover vehicle shown in the sketch was used to explore the surface of Mars as part of the Pathfinder mission in 1997. Use the data in the tables below to answer the questions that follow.

Determine answers for tests done on earth

Sojourner Data

Mass of Sojourner vehicle: 11.5kg

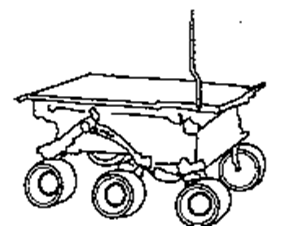
Wheel diameter: 0.13 m

Stored energy available: 5.4 x l05 J

Power required for driving under average conditions: 10 W

Land speed: 6.7 x 10-3 rn/s

We will calculate the answers based on the thing being on the earth.



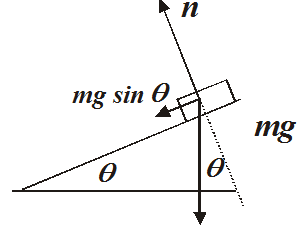
1. Assume that when leaving the Pathfinder spacecraft Sojourner rolls down a ramp inclined at 20° to the horizontal. The ramp must be lightweight but strong enough to support Sojourner. Calculate the minimum normal force that must be supplied by the ramp.
2. What is the net force on Sojourner as it travels across the earth’s surface at constant velocity? Justify your answer.
3. Determine the maximum distance that Sojourner can travel on a horizontal earth’s surface using its stored energy.
4. Suppose that 0.010% of the power for driving is expended against atmospheric drag as Sojourner travels on the surface. Calculate the magnitude of the drag force.

***Solution:***

1. Draw a FBD:

,

.



2. Since the thing is moving at a constant velocity, the sum of the forces is zero. Therefore the net force acting on the thing is zero.

3. ,

.

4. Suppose that 0.010% of the power for driving is expended against atmospheric drag as Sojourner travels on the surface. Calculate the magnitude of the drag force.

  ,

.

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**Lecture 8**

**Lecture topic: Work and Energy Theorem.** **Conservative Forces and Potential Energy**

**The plan**

1. Basic equations

2. Work and Energy Theorem

3. Conservative Forces and Potential Energy

4. Examples

**1. Basic equations**

Here are the equations that you will have available for you on tasks.

.

This is *the equation for kinetic energy.*

.

This is your *basic equation for gravitational potential energy.*

.

*The equation for work.*

Remember that if the angle θ is zero, the equation simplifies and becomes the first one. The Δ*r* is the displacement.

*Power equation.* Here it’s called average power.

,

.

*Another power equation.* This here one uses an applied force and the velocity of the object. The  part is for when the velocity is at an angle to the direction of the force.

.

*Hooke’s law*, the force on a spring deal. It gives the force required to compress a spring. This is also the force that the compressed spring can exert when it is released. The minus signs just means that the force the spring exerts is in the opposite direction of the force that compressed the spring.

.

*Kinetic energy stored in a spring.*

These are the equations for mechanical work, power, and energy. Later on in the course there will be other equations for energy as well, especially electrical potential energy and the maximum kinetic energy an electron can have after its been knocked out of the surface of a metal (the old photoelectric effect). But rest easy, we’ll get there.

Here is a list of the different sorts of things that you should be able to do in order to demonstrate your mastery of the material.

**2. Work and Energy Theorem**

1. You should understand the definition of work so you can:
2. Calculate the work done by a specified constant force on a body that undergoes a specified displacement.

*Use the equation for work.*

1. Relate the work done by a force to the area under a graph of force as a function of position and calculate this work in the case where the force is a linear function of position.

This is where you have a graph with force on the y axis and displacement on the x axis. The area under the curve represents the work done in changing an objects displacement from one point to another. We did a couple three of these problems. Kind of a geometry thing to find the area under the curve if the force isn’t constant. Of course if the force is constant then the area is simply the width times the height and the work is F times d.

1. Use the scalar product operation to calculate the work performed by a specified constant force ***F*** on a body that undergoes a displacement in a plane.

*The “scalar product operation” sounds pretty hairy, but actually means that you “multiply”. Anyway, you just use you the good old work equation.*

1. You should understand the work-energy theorem so you can:
2. Calculate the change in kinetic energy or speed that results from performing a specified amount of work on a body.

The Physics Kahuna worked hard to drill this into your brains. The basic idea is that the change in kinetic energy of a system is equal to the work done on the thing. This is also true for the change in potential energy for a system. It takes work to change something’s potential energy. It also takes work to change its kinetic energy. If you do 45 J of work changing an object’s potential energy, it then has 45 J of potential energy. This means it can then do 45 J of work. This work can be transformed into other forms of energy – kinetic, thermal, etc.

1. Calculate the work performed by the net force, or by each of the forces that makes up the net force, on a body that undergoes a specified change in speed or kinetic energy.

The idea here is that the work done on the body is equal to its change in kinetic energy. So if you know the change in speed of the thing you can find its change in kinetic energy which is equal to the work done, etc. The force is involved usually to find the acceleration of the system. Once you know the acceleration you can find the speeds, and once you know the speeds, you can find the change in kinetic energy.

1. Apply the theorem to determine the change in a body’s kinetic energy and speed that results from the application of specified forces, or to determine the force that is required in order to bring a body to rest in a specified distance.

This is a lot like the item above. Once again you use the force to find the acceleration and then that gets you into the whole speed/kinetic energy thing. You can also work backwards – kinetic energy to change in speed to acceleration to force.

**3. Conservative Forces and Potential Energy**

1. You should understand the concept of conservative forces so you can:

1. Write an expression for the force exerted by an ideal spring and for the potential energy stored in a stretched or compressed spring.

You just write out the equations, which are given. Easy as pie.

2. Calculate the potential energy of a single body in a uniform gravitational field.

Use the ** equation.

1. You should understand conservation of energy so you can:

1. Identify situations in which mechanical energy is or is not conserved.

Energy is always conserved. Mechanical energy though means potential energy and kinetic energy. The main types of potential energy would be gravitational, energy in a spring. Later there will also be potential energy from an electric field, potential energy stored in a capacitor, and potential energy of photoelectric electrons. It’s all treated the same.

*In elastic collisions we assume that kinetic energy is conserved.*

*Examples where mechanical energy is not conserved is when you have friction involved. The frictional force does work, which is an energy loss. Basically you have this:*

*.*

*We did several problems involving this sort of thing. Mainly with objects sliding down ramps where there was a coefficient of friction. You recall you had stuff like;*

*,*

*where is work done by friction – friction force multiplied by displacement. Here d is the distance the thing slid.*

2. Apply conservation of energy in analyzing the motion of bodies that are moving in a gravitational field and are subject to constraints imposed by strings or surfaces.

*Think of things swinging off a platform type deal from a string to some lower height. We did several of these. These are your Tarzan on a grapevine type deal. Also you could get an object moving from a table top to the deck below.*

*In both of these examples, the body would undergo a change in potential energy. You’d usually have to find out what its new velocity would be or what the change in height was – that kind of stuff.*

3. Apply conservation of energy in analyzing the motion of bodies that move under the influence of springs.

*Did you not totally love the spring energy problems we did? Go look at ‘em and relive the pleasure.*

*Power.* You should understand the definition of power so you can:

(1) Calculate the power required to maintain the motion of a body with constant acceleration (e.g., to move a body along a level surface, to raise a body at a constant rate, or to overcome friction for a body that is moving at a constant speed).

*Use the*

**

*equation. Use the acceleration to find F, which will be the net force, i.e., the sum of the forces. If they throw friction at you or some other force, just remember that the force you find using acceleration is the net force. You’ll have to write an equation for the sum of the forces.*

(2) Calculate the work performed by a force that supplies constant power, or the average power supplied by a force that performs a specified amount of work.

*Use the*

**

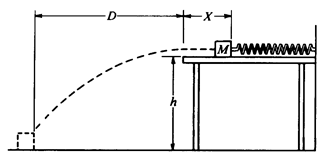
*equation as above. To get into work, use the general work equation. Just remember that you’re dealing with the net force as above.*

The big thing on the test will be conservation of energy – the idea that the energy before has to equal the energy after. You should know when and how to use this concept. Expect spring problems or gravity problems, but there could be other ways to sore energy as well. The concepts will commonly be folded in with other stuff you haven’t had yet as well – electricity, magnetism, or nuclear physics for example.

Conservation of energy is one of the biggest deals in physics, so be really good at it because it will be all over the test. Here are a couple of typical problems off previous tests/

**4. Examples**

* ***Example 1.*** One end of a spring is attached to a solid wall while the other end just reaches to the edge of a horizontal, frictionless tabletop, which is a distance ***h***above the floor. A block of mass ***M*** is placed against the end of the spring and pushed toward the wall until the spring has been compressed a distance ***X***, as shown below. The block is released, follows the trajectory shown, and strikes the floor a horizontal distance ***D***from the edge of the table. Air resistance is negligible.



Determine expressions for the following quantities in terms of ***M***, ***X***, ***D***, ***h***,and ***g***. Note that these symbols do not include the spring constant.

* 1. The time elapsed from the instant the block leaves the table to the instant it strikes the floor.

*This time is controlled by the time it takes for the block to fall.*

,

.

*(This is a projectile motion problem, ain’t it?)*

* 1. The horizontal component of the velocity of the block just before it hits the floor.

*Velocity is constant in the x direction. We’ve figured out the time of flight.*

.

The work done on the block by the spring.

*Let’s use conservation of energy to solve this one. Finally we get into work and energy.*

.

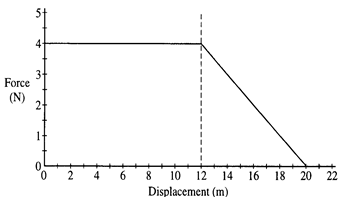
* 1. The spring constant.

*The work is also equal to the potential energy of the spring.*

.

Here’s another lovely problem:

* ***Example 2.*** A 0.20 *kg* object moves along a straight line. The net force acting on the object varies with the object's displacement as shown in the graph. The object starts from rest at displacement *x* = 0 and time *t* = 0 and is displaced a distance of 20 *m*. Determine each of the following.



a. The acceleration of the particle when its displacement ***x*** = 6 *m*.

*The force is constant from time zero till its displacement is 6 meters, so we can use the second law.*

.

b. The time taken for the object to be displaced the first 12 *m*.

*Again, the force is constant from the start till the displacement is 12 m. This means that the acceleration is also constant, so we can use one of the kinematic equations to find the time.*

.

c. The amount of work done by the net force in displacing the object the first 12 *m*.

*The work done is the area under the curve:*

*.*

d. The speed of the object at displacement ***x*** = 12 *m*.

*We can use conservation of energy to solve this bit.*

.

e. The final speed of the object at displacement

***x*** = 20 *m*.

We can use conservation of energy again. The work is equal to the area under the curve, so we add the area of the rectangle to the area of the triangle.

.

f. The change in the momentum of the object as it is displaced from ***x*** = 12 *m* to ***x***=20 *m.*

***NASA's Foam Test in Kinetics***

NASA's Foam Test Offered a Vivid Lesson in Kinetics.

HOUSTON, June 4 - The recent test suggesting that falling foam at liftoff could have caused the damage that doomed the space shuttle Columbia was a jaw-dropping demonstration of the destructive power of something so light, a NASA official said today.



Figure 3. Columbia Accident Investigation BoardIn this frame from film of a test, foam is seen after it hit a mock-up space shuttle wing at great speed and shatters, leaving V-shaped tracks.

"I thought: `Oh, my God! This is something. This isn't just a light bounce,' " recalled the official, G. Scott Hubbard, the director of the Ames Research Center at NASA and also a member of the independent board investigating the disaster.

Mr. Hubbard watched the test last Thursday at the Southwest Research Institute in San Antonio, and described it in detail to reporters in a briefing here this morning as a prelude to further tests on Thursday.

Before last week's test, many engineers at NASA said they thought lightweight foam could not harm the seemingly tough composite panels, and privately predicted that the foam would bounce off harmlessly, like a Nerf ball. But Mr. Hubbard said the experiment showed that "people's intuitive sense of physics is sometimes way off."

In last week's experiment, the researchers shot a 1.7-pound piece of foam at a mock-up shuttle wing at 531 miles per hour, roughly the speed of the chunk of foam that hit the Columbia wing about 81 seconds after liftoff.

Film of the experiment, released today, shows that the impact of a piece of foam hitting the wing mock-up caused the leading-edge panel to ripple like the surface of a struck gong.

The foam shattered, with hunks cramming their way into the seam between the panel and an adjoining seal. That opened a long slit in the surface of the wing four-tenths of an inch wide and about 22 inches long — potentially, more than enough to let in the stream of superheated gases that melted the wing from the inside out as the craft entered the atmosphere on Feb. 1.

Even the researchers setting up the test were unprepared for the sheer force of impact as a wave of energy moved through the inner structure of the wing and sideways along its panels — in some places, with seven times the force that the researchers had expected. Sensors inside the wing were knocked loose.

Bouncing a small piece of foam lightly between his hands for emphasis, Mr. Hubbard said: "You don't feel this can do anything. But you fire this at 500 miles an hour, and you saw it."

He invoked the physics equation that describes the amount of kinetic energy in a moving object, saying, "That's when it came home to me what (1/2) mv2 means." The simple equation says that kinetic energy is one-half times an object's mass times the object's velocity squared, so that even something very light can carry a great deal of force if it is moving fast enough. In fact, he said, the force was equivalent to catching a basketball thrown at 500 miles per hour.

Later analysis of the test panels showed that the stress from the impact shifted the struck panel to the right by one and a half inches, and that the seal, called a T-seal for its shape, was permanently deformed by one-tenth of an inch even after the foam had been removed.

The exact conditions of the actual foam strike — with extremes of vibration and temperature and near vacuum, could not be duplicated at the test site, so the researchers have had to improvise and try to match the conditions as best they can, Mr. Hubbard said.

Saying that he spoke only for himself and not the investigation board, Mr. Hubbard said that although the experiment "moves us a lot closer to saying that foam can do this kind of damage," it did not rule out other possible causes of the hole in the wing, including small meteorites and debris in space.

At Cape Canaveral, Fla., the chief of the NASA team that is collecting and examining debris from the Columbia said today that its analysis was consistent with that of the independent investigation board. "We have proven, based on the debris alone, where the breach was," said the official, Michael D. Leinbach, who is also the shuttle launching director.

Following the same guidelines used by the National Transportation Safety Board in aviation accidents, he said, the team analyzed the debris separately from any other data gathered by recorders and other sources, and determined that the shuttle was doomed by a breach at the bottom of Panel 8 on the left wing's leading edge.

The next round of tests in Texas could add weight to the growing consensus about the cause of the accident. Last week's tests used wing panels from the Enterprise, a test vehicle that never flew in space. That craft's leading edge panels were made from fiberglass because the Enterprise never had to face the heat of re-entry.

Foam testing will resume on Thursday with the first effort to fire a chunk of foam at the actual material used on the leading edge of the shuttle's wing. The material, reinforced carbon-carbon taken from the shuttle Discovery, is substantially weaker and less flexible than fiberglass.

The researchers estimate that the test will exert about 70 percent more force than necessary to shatter a composite panel, Mr. Hubbard said. "Now, whether it actually turns out that way or not, that's why we do the experiment," he said. "But the analysts are saying it looks like it'll break it."

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